Sauer–Shelah–Perles Lemma for Lattices

Joint work with
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VC dimension

The VC dimension of a family $\mathscr{F} \subseteq \{0,1\}^X$ is the maximal size of a shattered set.

X

$\mathscr{F}$

1	1	0	0	0
0	1	1	0	0
0	0	1	1	0
0	0	0	1	1

Shattered

VC dimension = 2

VC dimension

Relation to learning:
Hypothesis class is *PAC-learnable* iff it has finite VC dimension.

Sauer–Shelah–Perles lemma:
If $\mathcal{F} \subseteq \{0,1\}^X$ has VC dimension d then $|\mathcal{F}| \leq \binom{|X|}{\leq d}$.

Dichotomy theorem:
Let $\mathcal{F} \subseteq \{0,1\}^X$, where X is infinite.
If $\mathrm{VC}(\mathcal{F}) < \infty$ then $|\mathrm{proj}(\mathcal{F}, S)| \leq \mathit{poly}(|S|)$ for all $S \subseteq X$.
If $\mathrm{VC}(\mathcal{F}) = \infty$ then $|\mathrm{proj}(\mathcal{F}, S)| = 2^{|S|}$ for infinitely many S.

q-analog of VC dimension

Can we define VC dimension for families of subspaces over some finite field $\mathbb{F}$?

Alternative definition of VC dimension for sets:
The VC dimension of family $\mathscr{F} \subseteq 2^X$ is the maximum size of a shattered set.
A family $\mathscr{F} \subseteq 2^X$ shatters a set $S \subseteq X$ if $S \cap \mathscr{F}$ consists of all subsets of S.

	1	2	3	4	5	Intersection with $\{2,3\}$
{1,2}	1	1	0	0	0	$\{2\}$
{2,3}	0	1	1	0	0	$\{2,3\}$
{3,4}	0	0	1	1	0	$\{3\}$
{4,5}	0	0	0	1	1	$\varnothing$

q-analog of VC dimension

Alternative definition of VC dimension for sets:
The VC dimension of family $\mathscr{F} \subseteq 2^X$ is the maximum size of a shattered set.
A family $\mathscr{F} \subseteq 2^X$ shatters a set $S \subseteq X$ if for $S \cap \mathscr{F}$ consists of all subsets of S.

Definition of VC dimension for vector spaces
The VC dimension of family $\mathscr{F}$ of subspaces of $\mathbb{F}^n$ is the maximum dimension of a shattered subspace.
A family $\mathscr{F}$ shatters a subspace S of $\mathbb{F}^n$ if $S \cap \mathscr{F}$ consists of all subspaces of S.

Sauer–Shelah–Perles lemma [Babai–Frankl]:
If $\mathscr{F}$ is a family of subspaces of $\mathbb{F}^n$ that has VC dimension d then $|\mathscr{F}| \leq \begin{bmatrix} n \\ \leq d \end{bmatrix}_{|\mathbb{F}|}$.

Proving the Sauer–Shelah–Perles lemma

Sauer–Shelah–Perles lemma:
If $\mathscr{F} \subseteq \{0,1\}^X$ has VC dimension d then $|\mathscr{F}| \le \binom{|X|}{\le d}$.

Pajor's strengthening:
If $\mathscr{F} \subseteq \{0,1\}^X$ then $\mathscr{F}$ shatters at least $|\mathscr{F}|$ many sets.

Method 1: Induction on $|X|$.
Decompose $\mathscr{F} = \{S \in \mathscr{F} : x \in S\} \cup \{S \in \mathscr{F} : x \notin S\}$ for an arbitrary $x \in X$.

Method 2: Monotonization.
Lemma trivial for downward-closed families.
Monotonization increases number of shattered sets.

Method 3: Polynomial / linear algebra method.

Linear algebra proof

Pajor's strengthening:
If $\mathcal{F} \subseteq \{0,1\}^X$ then $\mathcal{F}$ shatters at least $|\mathcal{F}|$ many sets.

Proof idea:
Every function $\mathcal{F} \to \mathbb{R}$ can be expressed as linear combination of monomials corresponding to shattered sets.

Key observation:
If $\mathcal{F}$ does not shatter S then x_S is expressible as linear combination of smaller monomials *for inputs in* $\mathcal{F}$.

Proof by example:

- If $\{1,2\} \notin \mathcal{F} \cap \{1,2\}$ then $x_1 x_2 = 0$.
- If $\{1\} \notin \mathcal{F} \cap \{1,2\}$ then $x_1 x_2 = x_1$.
- If $\varnothing \notin \mathcal{F} \cap \{1,2\}$ then $x_1 x_2 = x_1 + x_2 - 1$.

Extends to vector spaces!

Sauer–Shelah–Perles lemma for lattices

Proof works for any lattice of flats in a matroid (*geometric lattice*).

- Complete uniform matroid: usual SSP lemma.
- Complete linear matroid: SSP lemma for vector spaces.
- Complete graphical matroid: SSP lemma for partitions.

More generally, proof holds whenever the Möbius function doesn't vanish.

- If $\{1,2\} \notin \mathscr{F} \cap \{1,2\}$ then $x_1 x_2 = 0$.
- If $\{1\} \notin \mathscr{F} \cap \{1,2\}$ then $x_1 x_2 = 1 \cdot x_1$.
- If $\varnothing \notin \mathscr{F} \cap \{1,2\}$ then $x_1 x_2 = 1 \cdot x_1 + 1 \cdot x_2 - 1$.

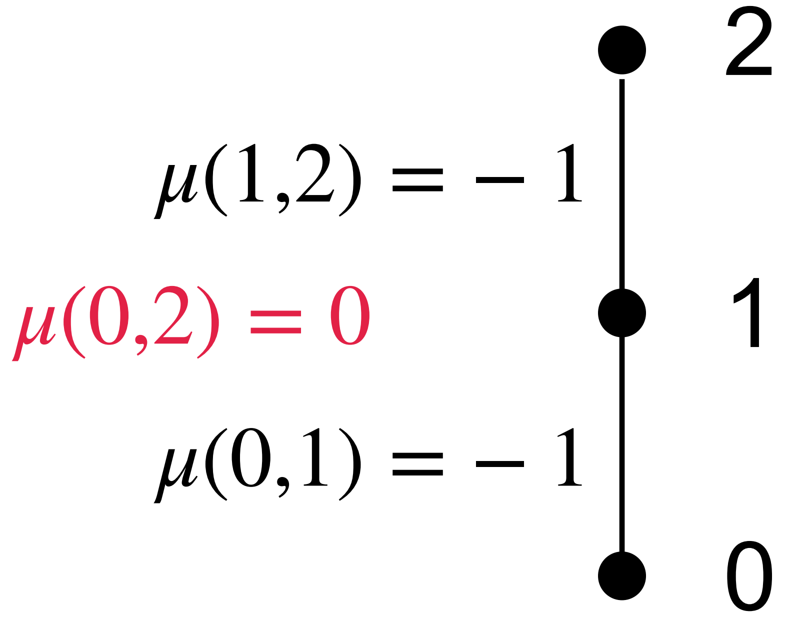
Negated Möbius function

When does Sauer–Shelah–Perles lemma hold?

Sauer–Shelah–Perles lemma for lattice $\mathscr{L}$:
If $\mathscr{F} \subseteq \mathscr{L}$ then $\mathscr{F}$ shatters at least $|\mathscr{F}|$ many elements of $\mathscr{L}$.

Babai–Frankl: SSP holds for $\mathscr{L}$ if $\mu(x, y) \neq 0$ for all $x \leq y$.

SSP doesn't hold:
$\{1,2\}$ only shatters 0



$2 \wedge \{1,2\} = \{1,2\}$

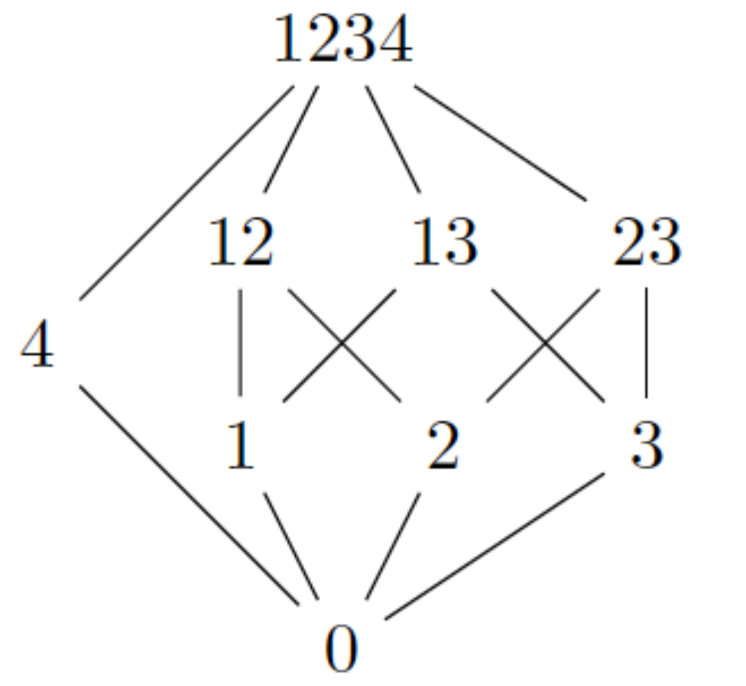
$1 \wedge \{1,2\} = \{1\}$

$0 \wedge \{1,2\} = \{0\}$

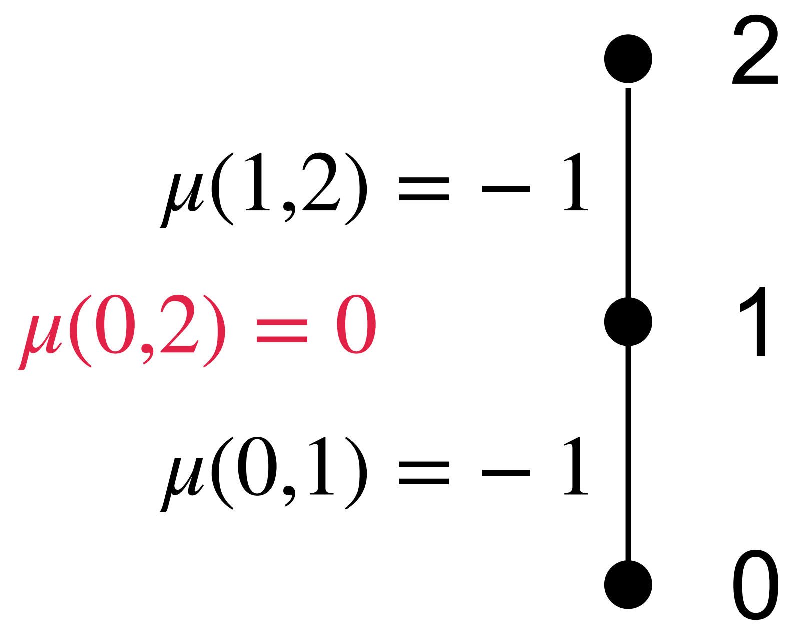
When does Sauer–Shelah–Perles lemma hold?

Babai–Frankl: SSP holds for $\mathscr{L}$ if $\mu(x, y) \neq 0$ for all $x \leq y$.

SSP holds for some lattices with vanishing Möbius function:



Doesn't hold for 3-element interval:



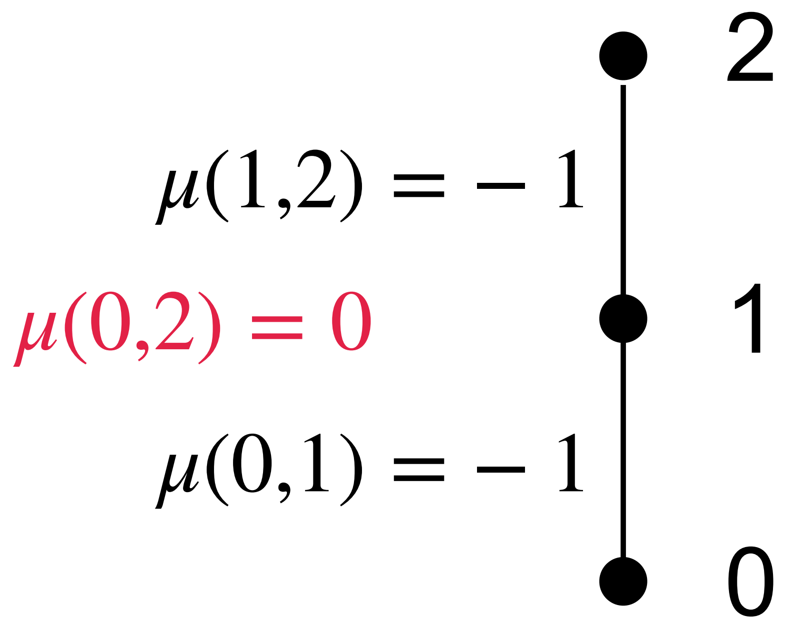
Doesn't hold if lattice *contains* 3-element interval, i.e., points $x < z$ with exactly one solution to $x < y < z$.

Conjecture: SSP holds *iff* lattice contains no 3-element interval (lattice is *relatively complemented*).

Relative complementation

Lattice is *relatively complemented* if for every $x < y < z$ there exists y' such that $y \wedge y' = x$ and $y \vee y' = z$.

Doesn't hold for 3-element interval:



No y' satisfies $1 \wedge y' = 0$ and $1 \vee y' = 2$.

Björner: A lattice is relatively complemented iff it doesn't contain a 3-element interval.

Partial results

Conjecture: SSP holds iff lattice is relatively complemented (RC).

Babai and Frankl: If Möbius function never vanishes, lattice is SSP.

Theorem 1: If lattice is RC and $\mu(x, y) = 0$ only if x, y are minimal and maximal elements, then lattice is SSP.

Theorem 2: Product of SSP lattices is SSP.

Theorem 3: If lattice is RC then SSP holds for all families whose set of non-shattered elems contains a minimum.

Thanks!