Property Testing meets Universal Algebra: Oligarchy testing

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Introduction

The accused should be convicted if they have both the means and the motive.
Here is what the three judges had to say:

	Means	Motive	Guilty
Holmes	Yes	No	No
Brandeis	No	Yes	No
Cardozo	Yes	Yes	Yes
Majority	Yes	Yes	No

Oops!

Introduction

- This shows that Majority is not *admissible* for AND.
- A judgment aggregation function f: $\{0,1\}^n \longrightarrow \{0,1\}$ is *admissible for AND* if for all $x,y \in \{0,1\}^n$, we have $f(x \wedge y) = f(x) \wedge f(y)$.
- Which functions are admissible?
  - Dictators: $f(x) = x_i$
  - Constants: $f(x) = 0$, $f(x) = 1$
  - Oligarchies (ANDs): $f(x) = x_1 \wedge \cdots \wedge x_m$

Introduction

Theorem: ANDs and constants are only functions admissible for AND.

Are there other solutions which are admissible whp?
(i.e., $\Pr[f(x \wedge y) = f(x) \wedge f(y)] \approx 1$)

Theorem (Nehama): If f is approx admissible, it is approx an AND:

$$\Pr[f(x \wedge y) = f(x) \wedge f(y)] \geq 1-\varepsilon \implies f \text{ is } O(n\varepsilon)\text{-close to an AND}$$

Want to remove dependence on *n*!

Arrow's theorem

An election is being held using ranked ballots. The outcome has to be a ranking as well. The final relative ranking of two candidates should depend only on the voters' relative rankings of these two candidates (IIA).

	Order	A>B?	B>C?	C>A?
Anthony	A>C>B	Yes	No	No
Brutus	B>A>C	No	Yes	No
Caesar	C>B>A	No	No	Yes
Majority	???	No	No	No

Oops!

Linearity testing

The patient should be declared sane if the sandwich has chocolate or pickles, *but not both*.
Here is what three psychiatrists had to say, based on their observations:

	Chocolate	Pickles	Sane
Feurd	Yes	No	Yes
Adler	No	Yes	Yes
Lacan	Yes	Yes	No
Majority	Yes	Yes	Yes

Oops!

Universal Algebra

- In universal algebra, a function admissible for AND is called an *AND polymorphism*.

- Similarly, a function admissible for Arrow is an *NAE* polymorphism (NAE = Not All Equal), and a function admissible for linearity testing is an XOR polymorphism.

- Only polymorphisms of NAE are dictators.

- Only polymorphisms of XOR are XORs.

Universal Algebra

- A set of allowed rows is called *truth-functional* if the last column is a function of the previous ones, and this is the only constraint.
    - Both AND and XOR are truth-functional. NAE isn't.
- Dokow and Holzman showed that in the binary truth-functional setting, AND and XOR (on any number of inputs) are the only interesting cases.
    - In all other cases, the only polymorphisms are dictators and, sometimes, constants.

Schaefer's theorem

- If P≠NP then there are NP-intermediate problems (Ladner's theorem, proved by diagonalization). Yet most problems we encounter in real life are either in P or are NP-hard.
- Schaefer's theorem states that this is the case for all CSPs (constraint-satisfaction problems): for each type of allowed constraints, the problem is either easy (in P) or hard (NP-complete).
  - 3SAT corresponds to the constraints $x \vee y \vee z$, with possibly negated inputs (eight possible constraints).
  - 3XOR-SAT corresponds to $x \oplus y \oplus z$ and its negation. Easy!
- Many generalizations: optimization problems, non-binary domains.

Property Testing

- You are giving me a function f: $\{0,1\}^n \longrightarrow \{0,1\}$ as a black box (think D-Wave), and claiming that f is an XOR (“linear”). I want to test this by querying the function at only a few places.

- Natural test: pick x,y at random, and verify $f(x \oplus y)=f(x) \oplus f(y)$.

- If f is linear, test always passes (“completeness”).

- If test passes w.p. $1–\varepsilon$, f is $O(\varepsilon)$-close to an XOR (“soundness”).

  - Note no dependence on n. In other cases (e.g. monotonicity testing), dependence on n is necessary.

Linearity testing

How do we prove soundness?

- Method 1: Self-correction
  - For most x,y: $f(x) = f(y) \oplus f(x \oplus y)$.
  - “Guess” correct value at x is majority of $f(y) \oplus f(x \oplus y)$.
  - BLR: This works for ε<const!
- Method 2: Fourier analysis
  - Express success probability of test using Fourier expansion of f.

Fourier analysis

- Change notation to $f\colon \{-1,1\}^n \longrightarrow \{-1,1\}$.
- f can be expressed uniquely as a multilinear polynomial.
- Each monomial is an XOR of a subset $S \subseteq [n]$ of variables.
  - Denote coefficient by $\hat{f}(S)$ ("Fourier coefficient").
- $\Pr[f(xy)=f(x)f(y)] = ½+½\sum\hat{f}(S)^3$.
- If $\Pr[f(xy)=f(x)f(y)] \approx 1$ then some Fourier coefficient is close to 1.
  - f is close to the corresponding XOR.

Oligarchy testing

Given f: $\{0,1\}^n \longrightarrow \{0,1\}$ s.t. $f(xy) = f(x)f(y)$ whp, want to deduce that f is close to an AND.

- Method 1: Self-correction
  - Cannot express $f(x)$ in terms of $f(y),f(xy)$.
    "Information is lost."
- Method 2: Fourier analysis
  - Formula for $\Pr[f(xy) = f(x)f(y)]$ isn't nice any more.
    For linearity testing, lucky that XORs=monomials.

Our approach

Suppose $f(xy) = f(x)f(y)$ w.p. ≈ 1.

- Fix x, and take expectation over y:
  - $T_{\downarrow}f(x) \approx \lambda f(x)$, where $\lambda = \mathbb{E}[f]$.
  - $T_{\downarrow}f(x)$ is average of $f(z)$ on all values $z \leq x$.
- In total, $T_{\downarrow}f \approx \lambda f$ (in appropriate norm).
  - So need to determine approximate eigenvectors of $T_{\downarrow}$.

Our approach

- $T_\downarrow$ is one-sided variant of more familiar noise operator:
  - $Tf(x)=\mathbb{E}[f(x\oplus y)]$, where y is biased.
- Eigenvectors of T are XORs; form an orthogonal basis.
  - Implies that approx eigenvectors are close to eigenvectors.
- In contrast, eigenvectors of $T_\downarrow$ are ANDs; not orthogonal!
  - Same approach cannot work.

Some examples

$$f(x) = \begin{cases} x_1 \vee x_2 & \text{if } |x| \geq n/3 \\ x_1 \oplus x_2 & \text{if } |x| < n/3 \end{cases}$$

For random x,y, $|x| \geq n/3$ while $|x \wedge y| < n/3$, so:

- $f(x) = x_1 \vee x_2$ while $T_\downarrow f(x) \approx \mathbb{E}[(x_1 \wedge y_1) \oplus (x_2 \wedge y_2)]$
- If $x_1 = x_2 = 0$ then $x_1 \wedge y_1 = x_2 \wedge y_2 = 0$, so $f(x) = 0$ and $T_\downarrow f(x) \approx 0$. (In fact, $T_\downarrow f(x) = 0$.)
- If (e.g.) $x_1 = 1$ then $x_1 \wedge y_1 = y_1$ is a random bit, so $f(x) = 1$ and $T_\downarrow f(x) \approx 1/2$.
- In total, $T_\downarrow f \approx ½f$.

$$f(x) = \begin{cases} 1 & \text{if } |x| \geq n/3 \\ \mathrm{Ber}(\lambda) & \text{if } |x| < n/3 \end{cases}$$

This time, $T_\downarrow f \approx \lambda \approx \lambda f$.

Some examples

$$g(x) = x_1 \vee x_2$$

$$f(x) = x_1 \oplus x_2$$

For *all* x,y:

- $g(x) = x_1 \vee x_2$ while $T_\downarrow f(x) = \mathbb{E}[(x_1 \wedge y_1) \oplus (x_2 \wedge y_2)]$
- If $x_1 = x_2 = 0$ then $x_1 \wedge y_1 = x_2 \wedge y_2 = 0$, so $g(x) = 0$ and $T_\downarrow f(x) = 0$.
- If (e.g.) $x_1 = 1$ then $x_1 \wedge y_1 = y_1$ is a random bit, so $g(x) = 1$ and $T_\downarrow f(x) = 1/2$.
- In total, $T_\downarrow f = \frac{1}{2} g$.

$$g(x) = 1$$

$$f(x) = \lambda$$

This time, $T_\downarrow f = \lambda = \lambda g$.

Generalized eigenfunctions

- It turns out that we will need to solve the following "generalized eigenfunction problem":
  - $T_\downarrow f = \lambda g$, where g: $\{0,1\}^n \longrightarrow \{0,1\}$ and f: $\{0,1\}^n \longrightarrow [0,1]$.
- The solution is a generalization of both examples:
  - g is an AND of disjoint ORs.
  - f is an AND of disjoint XORs (on same variables), multiplied by the appropriate constant factor.
- Proof is a nice combinatorial exercise.

Generalized eigenfunctions

Solving $T_{\downarrow}f=\lambda g$:

- Step 1: *g* has to be monotone.
- Step 2: all minterms of *g* have same size.
- Step 3: minterms constitute “complete multipartite graph”.

Solving $T_{\downarrow}f\approx\lambda g$:

- Apply linear programming duality to get “robust” version of same conclusion.
- Exponential dependence on *n*.

Noise is low-pass filter

- Recall the Fourier expansion of a function.
- Contribution of degree *d* monomials constitutes "*d*'th level".
- Classical noise operator has diminishing effect on high levels.
- Same holds for $T_\downarrow$, with a caveat:
  It translates "skewed" Fourier expansion to classical Fourier expansion, while diminishing high levels.
- Upshot is that if $T_\downarrow f \approx \lambda g$ then g is concentrated on low levels.
  - This implies that g is close to a "junta" (depends on few coords).

Finishing the proof

Suppose $T_{\downarrow} f \approx \lambda g$.

- g is close to a junta G on variables J.
- Average f on fibers of J (with respect to appropriate distribution!) to obtain a function F such that $T_{\downarrow} F \approx \lambda G$.
- Apply robust characterization of generalized eigenfunctions.

Final result: same as robust characterization, but:

- No dependence on n.
- Bad dependence on ε (doubly exponential).

Finishing the proof

Suppose $f(xy)=f(x)f(y)$ with high probability.

- Then $T_{\downarrow}f \approx \lambda f$, where $\lambda = \mathbb{E}[f]$.
- Apply previous result.
  - Value of λ forces f to be an AND (rather than an AND-OR).

Open problems

1. Improve dependence on ε from double exp to poly.
2. Generalize to general truth-functional setting.
    - In all remaining cases, answer should be dictator.
    - Known for Arrow's theorem using Fourier analysis (Kalai).
3. "List-decoding" version:
    - What if $\Pr[f(x \wedge y) = f(x) \wedge f(y)]$ is better than random?
        - If $\Pr[f(x \oplus y) = f(x) \oplus f(y)] > \frac{1}{2}$ then f correlates with some XOR.