Monotone Submodular Optimization over a Matroid

Yuval Filmus and Justin Ward

5 October 2015

The problem

Given:

- Matroid $\mathfrak{m}$.
- Monotone submodular f on $\mathfrak{m}$, $f(\emptyset) = 0$.

Goal:

- Find base $S \in \mathfrak{m}$ maximizing $f(S)$.

Hardness and algorithms

Hardness:

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$ *even for uniform matroid*:
  - NP-hardness for Maximum Coverage (Feige)

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:
  - NP-hardness for Maximum Coverage (Feige)
  - Oracle hardness (Nemhauser, Fisher, and Wolsey)

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:
  - NP-hardness for Maximum Coverage (Feige)
  - Oracle hardness (Nemhauser, Fisher, and Wolsey)

Algorithms:

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:
  - NP-hardness for Maximum Coverage (Feige)
  - Oracle hardness (Nemhauser, Fisher, and Wolsey)

Algorithms:

- Uniform matroid: Greedy is $1 - 1/e$ approximation

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:
  - NP-hardness for Maximum Coverage (Feige)
  - Oracle hardness (Nemhauser, Fisher, and Wolsey)

Algorithms:

- Uniform matroid: Greedy is $1 - 1/e$ approximation
- General matroid: Greedy is $1/2$ approximation

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:
  - NP-hardness for Maximum Coverage (Feige)
  - Oracle hardness (Nemhauser, Fisher, and Wolsey)

Algorithms:

- Uniform matroid: Greedy is $1 - 1/e$ approximation
- General matroid: Greedy is $1/2$ approximation
- Vondrák et al.: Continuous Greedy is $1 - 1/e$ approximation

Hardness and algorithms

Hardness:

- Generalizes classical NP-hard problem *Maximum Coverage*
- Hard to approximate better than $1 - 1/e$
  even for uniform matroid:
  - NP-hardness for Maximum Coverage (Feige)
  - Oracle hardness (Nemhauser, Fisher, and Wolsey)

Algorithms:

- Uniform matroid: Greedy is $1 - 1/e$ approximation
- General matroid: Greedy is $1/2$ approximation
- Vondrák et al.: Continuous Greedy is $1 - 1/e$ approximation

Is there an optimal combinatorial algorithm?

Local search

Algorithm:

$S \leftarrow$ Greedy
repeat
    find $x \in S, y \notin S$ s.t. $S - x + y \in \mathfrak{m}$ and $f(S - x + y) > f(S)$
    $S \leftarrow S - x + y$
until no such x, y found
return S

Local search

Algorithm:

$S \leftarrow$ Greedy
repeat
  find $x \in S, y \notin S$ s.t. $S - x + y \in \mathfrak{m}$ and $f(S - x + y) > f(S)$
  $S \leftarrow S - x + y$
until no such x, y found
return S

Gives $1/2$ approximation unless exchanging $\Omega(r)$ elements!

Local search

Algorithm:

$S \leftarrow$ Greedy
repeat
  find $x \in S, y \notin S$ s.t. $S - x + y \in \mathfrak{m}$ and $g(S - x + y) > g(S)$
  $S \leftarrow S - x + y$
until no such x, y found
return S

Gives $1 - 1/e$ approximation for some explicit g!

Local search

Algorithm:

$S \leftarrow$ Greedy
repeat
    find $x \in S, y \notin S$ s.t. $S - x + y \in \mathfrak{m}$ and $g(S - x + y) > g(S)$
    $S \leftarrow S - x + y$
until no such x, y found
return S

Gives $1 - 1/e$ approximation for some explicit g!

$$g(S) = \int_0^1 \frac{e^p \, dp}{e - 1} \sum_{T \subseteq S} p^{|T|-1} (1 - p)^{|S|-|T|} f(T).$$

Algorithm analysis

Lemma (Brualdi)

If $S, O \in \mathfrak{m}$, can enumerate

$$S = \{s_1, \ldots, s_r\}, \quad O = \{o_1, \ldots, o_r\}$$

so that $S - s_i + o_i \in \mathfrak{m}$ for all i.

Algorithm analysis

Lemma (Brualdi)

If $S, O \in \mathfrak{m}$, can enumerate

$$S = \{s_1, \ldots, s_r\}, \quad O = \{o_1, \ldots, o_r\}$$

so that $S - s_i + o_i \in \mathfrak{m}$ for all i.

- O — optimal solution of f, S — local optimum of g.

Algorithm analysis

Lemma (Brualdi)

If $S, O \in \mathfrak{m}$, can enumerate

$$S = \{s_1, \ldots, s_r\}, \quad O = \{o_1, \ldots, o_r\}$$

so that $S - s_i + o_i \in \mathfrak{m}$ for all i.

- O — optimal solution of f, S — local optimum of g.
- $g(S - s_i + o_i) \leq g(S)$ for all i.

Algorithm analysis

Lemma (Brualdi)

If $S, O \in \mathfrak{m}$, can enumerate

$$S = \{s_1, \ldots, s_r\}, \quad O = \{o_1, \ldots, o_r\}$$

so that $S - s_i + o_i \in \mathfrak{m}$ for all i.

- O — optimal solution of f, S — local optimum of g.
- $g(S - s_i + o_i) \leq g(S)$ for all i.

Lemma (Main)

$$\frac{e}{e-1} f(S) \geq f(O) + \sum_{i=1}^{r} [g(S) - g(S - s_i + o_i)] .$$

Finding g

Let

$$g(S) = \sum_{T \subseteq S} c(|S|, |T|)\, f(T).$$

Finding g

Let

$$g(S) = \sum_{T \subseteq S} c(|S|, |T|)\, f(T).$$

Solve

$$\begin{aligned} \max_c \min_f \; & f(S) \\ s.t. \; & f(O) = 1 \\ & g(S) \geq g(S - s_i + o_i) \, \forall i \end{aligned}$$

Finding g

Let

$$g(S) = \sum_{T \subseteq S} c(|S|, |T|)\, f(T).$$

Solve

$$\begin{aligned} \max_c \min_f \; & f(S) \\ s.t.\; & f(O) = 1 \\ & g(S) \geq g(S - s_i + o_i)\, \forall i \end{aligned}$$

Hint: dualize inner LP to get an LP.

More on g

Define $f_A(x) = f(A + x) - f(A)$.

$$g_A(x) = \int_0^1 \frac{e^p \, dp}{e - 1} \sum_{B \subseteq A} p^{|B|}(1 - p)^{|A|-|B|} f_B(x)$$

More on g

Define $f_A(x) = f(A+x) - f(A)$.

$$g_A(x) = \int_0^1 \frac{e^p\,dp}{e-1} \sum_{B\subseteq A} p^{|B|}(1-p)^{|A|-|B|} f_B(x)$$
$$= \int_0^1 \frac{e^p\,dp}{e-1} \underbrace{\phi(p,\ldots,p)}_{\substack{\text{continuous relaxation} \\ \text{of } B \mapsto f_B(x)}} .$$

More on g

Define $f_A(x) = f(A + x) - f(A)$.

$$g_A(x) = \int_0^1 \frac{e^p\,dp}{e-1} \sum_{B \subseteq A} p^{|B|}(1-p)^{|A|-|B|} f_B(x)$$

$$= \int_0^1 \frac{e^p\,dp}{e-1} \underbrace{\phi(p, \ldots, p)}_{\substack{\text{continuous relaxation} \\ \text{of } B \mapsto f_B(x)}} .$$

g is monotone and submodular!

More on $g_A(x)$

Suppose that $g_A(x) = \sum_{B \subseteq A} \gamma(|A|, |B|)\, f_B(x)$.

More on $g_A(x)$

Suppose that $g_A(x) = \sum_{B \subseteq A} \gamma(|A|, |B|)\, f_B(x)$.

- g monotone $\longleftrightarrow \gamma(|A|, |B|) \geq 0$.

More on $g_A(x)$

Suppose that $g_A(x) = \sum_{B \subseteq A} \gamma(|A|, |B|)\, f_B(x)$.

- g monotone $\longleftrightarrow$ $\gamma(|A|, |B|) \geq 0$.
- g submodular $\longleftrightarrow$ for some distribution Γ on sets and $C > 0$,

$$g_A(x) = C \mathop{\mathbb{E}}_{T \sim \Gamma}\left[f_{T \cap A}(x)\right].$$

More on $g_A(x)$

Suppose that $g_A(x) = \sum_{B \subseteq A} \gamma(|A|, |B|)\, f_B(x)$.

- g monotone $\longleftrightarrow$ $\gamma(|A|, |B|) \geq 0$.
- g submodular $\longleftrightarrow$ for some distribution Γ on sets and $C > 0$,

$$g_A(x) = C \mathop{\mathbb{E}}_{T \sim \Gamma} [f_{T \cap A}(x)] .$$

de Finetti's theorem: for some distribution P on $[0, 1]$,

$$g_A(x) = C \mathop{\mathbb{E}}_{p \sim P} \left[\sum_{B \subseteq A} p^{|B|} (1-p)^{|A|-|B|} f_B(x) \right] .$$

More on $g_A(x)$

Suppose that $g_A(x) = \sum_{B \subseteq A} \gamma(|A|, |B|)\, f_B(x)$.

- g monotone $\longleftrightarrow$ $\gamma(|A|, |B|) \geq 0$.
- g submodular $\longleftrightarrow$ for some distribution Γ on sets and $C > 0$,

$$g_A(x) = C \mathop{\mathbb{E}}_{T \sim \Gamma} [f_{T \cap A}(x)] \,.$$

de Finetti's theorem: for some distribution P on $[0, 1]$,

$$g_A(x) = C \mathop{\mathbb{E}}_{p \sim P} \left[\sum_{B \subseteq A} p^{|B|} (1 - p)^{|A| - |B|} f_B(x) \right] \,.$$

Proof shows cdf Π of P satisfies $\Pi' = \Pi \longrightarrow \Pi'(p) = \frac{e^p}{e-1}$.

More on $g_A(x)$

Suppose that $g_A(x) = \sum_{B \subseteq A} \gamma(|A|, |B|)\, f_B(x)$.

- g monotone $\longleftrightarrow$ $\gamma(|A|, |B|) \geq 0$.
- g submodular $\longleftrightarrow$ for some distribution Γ on sets and $C > 0$,

$$g_A(x) = C \mathop{\mathbb{E}}_{T \sim \Gamma} [f_{T \cap A}(x)] \,.$$

de Finetti's theorem: for some distribution P on $[0, 1]$,

$$g_A(x) = C \mathop{\mathbb{E}}_{p \sim P} \left[\sum_{B \subseteq A} p^{|B|} (1-p)^{|A|-|B|} f_B(x) \right] \,.$$

Proof shows cdf Π of P satisfies $\Pi' = \Pi \longrightarrow \Pi'(p) = \frac{e^p}{e-1}$.
Approximation ratio is $1/\Pi'(1) = 1 - 1/e$.

Coverage functions

Setup:

Coverage functions

Setup:

- universe U with weight function $w \geq 0$

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U
- $f(\{s_1, \ldots, s_r\}) = w(s_1 \cup \cdots \cup s_r)$.

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U
- $f(\{s_1, \ldots, s_r\}) = w(s_1 \cup \cdots \cup s_r)$.

$$g(\{s_1, \ldots, s_r\}) = \sum_x w(x)\, \alpha(\#i \text{ s.t. } x \in s_i).$$

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U
- $f(\{s_1, \ldots, s_r\}) = w(s_1 \cup \cdots \cup s_r)$.

$$g(\{s_1, \ldots, s_r\}) = \sum_x w(x)\, \alpha(\#i \text{ s.t. } x \in s_i).$$

$$\alpha(k) = \int_0^1 \frac{1-(1-p)^k}{p} \frac{e^p\, dp}{e-1}$$

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U
- $f(\{s_1, \ldots, s_r\}) = w(s_1 \cup \cdots \cup s_r)$.

$$g(\{s_1, \ldots, s_r\}) = \sum_x w(x)\, \alpha(\#i \text{ s.t. } x \in s_i).$$

$$\alpha(k) = \int_0^1 \frac{1-(1-p)^k}{p} \frac{e^p\, dp}{e-1} = \sum_{\ell=0}^{k-1} \ell! \left(1 - \frac{1}{e-1} \sum_{m=1}^{\ell} \frac{1}{m!}\right)$$

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U
- $f(\{s_1, \ldots, s_r\}) = w(s_1 \cup \cdots \cup s_r)$.

$$g(\{s_1, \ldots, s_r\}) = \sum_x w(x)\, \alpha(\#i \text{ s.t. } x \in s_i).$$

$$\alpha(k) = \int_0^1 \frac{1-(1-p)^k}{p} \frac{e^p\, dp}{e-1} = \sum_{\ell=0}^{k-1} \ell! \left(1 - \frac{1}{e-1} \sum_{m=1}^{\ell} \frac{1}{m!}\right) \sim \frac{\ln k}{e-1}.$$

Coverage functions

Setup:

- universe U with weight function $w \geq 0$
- matroid $\mathfrak{m}$ on subsets of U
- $f(\{s_1, \ldots, s_r\}) = w(s_1 \cup \cdots \cup s_r)$.

$$g(\{s_1, \ldots, s_r\}) = \sum_x w(x)\, \alpha(\# i \text{ s.t. } x \in s_i).$$

$$\alpha(k) = \int_0^1 \frac{1-(1-p)^k}{p} \frac{e^p\, dp}{e-1} = \sum_{\ell=0}^{k-1} \ell! \left(1 - \frac{1}{e-1} \sum_{m=1}^{\ell} \frac{1}{m!}\right) \sim \frac{\ln k}{e-1}.$$

First few values: $0, 1, 1.418, 1.672, 1.852, 1.991, \ldots$

Open questions

- Matroid intersection?

Open questions

- Matroid intersection?
- Deterministic algorithm?

Open questions

- Matroid intersection?
- Deterministic algorithm?
- Other uses of non-oblivious local search?