Triangle-intersecting families of graphs

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1 Simonovits–Sós conjecture

In 1938, Erdős, Ko and Rado proved the basic result known as the Erdős–Ko–Rado theorem: (curiously, the paper [2] was published only in 1961)

**Theorem (Erdős–Ko–Rado).** Suppose $k \leq n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family (any two sets intersect). Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. If $k < n/2$, then this bound is achieved only for dictators (families of the form $\{S \in \binom{[n]}{k} : i \in S\}$).

Their paper opened up an entire field in extremal combinatorics. One of the questions, asked by Simonovits and Sós [5] in 1976, concerned triangle-intersecting families. A collection $\mathcal{F} \subseteq 2^{K_n}$ of graphs on $n$ vertices is triangle-intersecting if the intersection of any two graphs contains some triangle. It will be convenient to measure such families using the measure $\mu(\mathcal{F}) = |\mathcal{F}|/2^{\binom{n}{2}}$. One way of constructing such a family is a triangle-junta: take a fixed triangle and all graphs containing it. Such a family contains $1/8$ of the graphs. Simonovits and Sós conjectured that this is the best that can be achieved, and furthermore triangle-juntas are the unique maximizers. Unfortunately, all they could prove was an upper bound of $1/2$, which follows from the fact that a graph and its complement cannot both be in the family.

Chung, Graham, Frankl and Shearer [1] were able to prove an upper bound of $1/4$, using Shearer’s lemma. The lemma states that if you project the family $\mathcal{F}$ into $m$ subsets $X_1, \ldots, X_m$ such that each element is covered exactly $k$ times, then

$$\mu(\mathcal{F}) \leq \sqrt[k]{\mu(\mathcal{F}_1) \cdots \mu(\mathcal{F}_m)},$$

where $\mathcal{F}_i$ is the projection to $X_i$, and the measure $\mu$ is normalized to be a probability measure on each of the sets. The idea is to take as the sets $X_i$ all complements of complete bipartite graphs. For each bipartite graph $G$, if we project $\mathcal{F}$ to $\overline{G}$ then we get an intersecting family, since every triangle contains an edge outside of $G$. Therefore $\mu(\mathcal{F}_i) \leq 1/2$, since $\mathcal{F}_i$ cannot contain both a graph and its complement. On the other hand, each edge appears in half the families, so $k = m/2$. Therefore $\mu(\mathcal{F}) \leq ((1/2)^m)^{2/m} = 1/4$.

The proof only used the fact that a triangle is not bipartite. It therefore applies for a larger class of families, non-bipartite-intersecting or odd-cycle-intersecting. We can also improve on the proof in another respect. Instead of considering intersecting families, we can consider agreeing families. These are families in which the condition for each pair $A, B$ of sets is applied not to the intersection $A \cap B$ but to the agreement $A \Delta B = \overline{A \Delta B}$, which is the set of positions on which both sets “agree”. For any bipartite $G$, if we project an odd-cycle-agreeing family to $\overline{G}$ then we get an agreeing family, and such families have measure at most $1/2$, for the same reason as above. So the bound $1/4$ applies even for odd-cycle-agreeing families.
In the rest of the talk, we prove the Simonovits–Sós conjecture for odd-cycle-agreeing families.

2 Hoffman’s bound

The basic idea is to use a spectral bound due to Hoffman [3]. The bound, which is a special case of the Lovász bound (better known as the $\theta$ function), was devised to bound the size of independent sets in graphs. In our case, the graph is the non-agreement graph of our problem: the vertices are the graphs on $n$ vertices, and the edges connect any two graphs which are not odd-cycle-agreeing. An independent set in this graph is the same as an odd-cycle-agreeing family.

Lemma (Hoffman’s bound). Let $A$ be a symmetric matrix indexed by the graphs on $n$ vertices such that (i) $A_{GH} = 0$ whenever $G, H$ are odd-cycle-agreeing, (ii) $A \mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the constant vector. For every odd-cycle-agreeing family $\mathcal{F}$, $\mu(\mathcal{F}) \leq \frac{-\lambda_{\min}}{1-\lambda_{\min}}$, where $\lambda_{\min}$ is the minimal eigenvalue of $A$.

Proof. Let $f$ be the characteristic vector of $\mathcal{F}$. We consider $f$ under the inner product $\langle g, h \rangle = \mathbb{E} g(x) h(x)$. Under this inner product, $\langle f, f \rangle = \mu(\mathcal{F}) \mathbf{1} = \mu(\mathcal{F}) \mathbf{1} + g$. By construction $g \perp \mathbf{1}$, and so $\|g\|^2 = \|f\|^2 - \mu(\mathcal{F})^2 \langle \mathbf{1}, \mathbf{1} \rangle = \mu(\mathcal{F})^2 - \mu(\mathcal{F})^2$. The conditions on $A$ imply that

$$0 = \langle f, A f \rangle = \mu(\mathcal{F})^2 \langle \mathbf{1}, \mathbf{1} \rangle + \langle g, A g \rangle \geq \mu(\mathcal{F})^2 - \lambda_{\min} \|g\|^2.$$ 

Substituting the value of $\|g\|^2$, we obtain $\mu(\mathcal{F})^2 \leq \lambda_{\min}(\mu(\mathcal{F})^2)$, and so $\mu(\mathcal{F}) \leq \lambda_{\min}(1 - \mu(\mathcal{F}))$. The lemma easily follows. □

Hoffman’s bound isn’t always tight, but in our case it is. How do we come up with the matrix $A$? The first idea is to use some symmetry. If $\mathcal{F}$ is an odd-cycle-agreeing family then so is $\mathcal{F} \oplus G$ given by $(\mathcal{F} \oplus G)(H) = \mathcal{F}(G \oplus H)$. We can do the same operation on the matrix $A$, by defining $A^{\oplus G}(H, K) = A(H \oplus G, K \oplus G)$. Since $(H \oplus G) \vee (K \oplus G) = (H \oplus G) \oplus (K \oplus G) = H \oplus K = H \vee K$, we see that $A^{\oplus G}$ satisfies condition (i) in Hoffman’s bound. It is easy to see that condition (ii) is also satisfied, and furthermore $\lambda_{\min}(A^{\oplus G}) = \lambda_{\min}(A)$. We can therefore consider $A' = \mathbb{E}_G A^{\oplus G}$. Clearly $\lambda_{\min}(A') \geq \lambda_{\min}(A)$, and so since $\lambda_{\min}/(1 - \lambda_{\min}) = 1 - 1/(1 - \lambda_{\min})$ is decreasing in $\lambda_{\min}$, replacing $A$ with $A'$ can only result in a better bound. The matrix $A'$, in turn, is symmetric, that is $A^{\oplus G} = A'$. A straightforward calculation shows that the Fourier characters $\chi_G(H) = (-1)^{|G \cap H|}$ are all eigenvectors of $A'$, and so constitute its eigenvectors (since they form a basis). Summarizing, without loss of generality we can conclude that $A$ has the Fourier characters as eigenvectors.

We can say even more. The space of $2 \times 2$ matrices whose eigenvectors are the Fourier characters is spanned by two matrices: the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the swapping matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As a linear operator, the first matrix leaves its input unaffected, and the second flips it. Taking tensor products, we obtain a basis $B_G(H) = G \oplus H$. When $G$ is the complement of a bipartite graph, $B_G$ satisfies the properties in Hoffman’s bound. Condition (ii) is easy to check. To verify condition (i), suppose that $H, K$ are odd-cycle-intersecting. Then $1_H B_G 1_K = 1_H 1_K \oplus G = [H = K \oplus G] \neq [H \vee K = \mathcal{G}] = 0$, since $\mathcal{G}$ is bipartite. Therefore all matrices of the form $\sum_G \mathcal{G} B_{\mathcal{G}}$ satisfy the conditions of Hoffman’s bound, where $G$ goes over all bipartite graphs, and the $\alpha_G$ sum to 1. A straightforward inductive argument proves the converse: these are all the matrices satisfying the conditions.

It remains to choose the coefficients $\alpha_G$. To that end, we should understand what the eigenvalues of $B_{\mathcal{G}}$ look like. The eigenvalues of $I$ are both 1, while $X$ has eigenvalues 1, $-1$ for its eigenvectors $\chi_0, \chi_1$ (here 1 is a dummy element). The matrix $B_{\mathcal{G}}$ can be thought of as a tensor product
of copies of $I$ and $X$, where a copy of $X$ is used for each edge in $\overline{G}$. Therefore the eigenvalue corresponding to $\chi_H$ is $\lambda_H = (-1)^{|H \cap \overline{G}|} = (-1)^{|H|}(-1)^{|H \cap \overline{G}|}$. The general matrix therefore has eigenvalues

$$\lambda_H = (-1)^{|H|} \sum_{G} \alpha_G(-1)^{|H \cap G|}.$$ 

Call a vector of eigenvalues (or spectrum) admissible if it can be written in this form, ignoring the condition that the $\alpha_G$ sum to 1. Straightforward induction shows that for each bipartite $G$ and each function $f: G \rightarrow \mathbb{R}$, the following spectrum is admissible: $(-1)^{|H|}f(H \cap G)$. Conversely, every feasible spectrum is a linear combination of functions of this form.

At this point, a flash of inspiration is needed. We consider the following process: take a random complete bipartite graph $G$, and let $q_K(H)$ be the probability that $H \cap G$ is isomorphic to $G$, and $q_k(H)$ be the probability that $|H \cap G| = k$. By taking the weighted average over all bipartite $G$, we see that $(-1)^{|H|}q_K(H)$ and $(-1)^{|H|}q_k(H)$ are both admissible spectra. We will be looking for an admissible spectrum of the following form:

$$(-1)^{|H|} \sum_k c_k q_k(H).$$

The intuition here is that for large $|H|$, this spectrum is close to 0, while for small $|H|$, we might have enough degrees of freedom to control its minimum. To that end, we consider the following table:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$q_0(H)$</th>
<th>$q_1(H)$</th>
<th>$q_2(H)$</th>
<th>$q_3(H)$</th>
<th>$q_4(H)$</th>
</tr>
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<tbody>
<tr>
<td>$\emptyset$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-$</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\triangle$</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1/16</td>
<td>1/4</td>
<td>3/8</td>
<td>1/4</td>
<td>1/16</td>
</tr>
<tr>
<td>$K_4^-$</td>
<td>1/8</td>
<td>0</td>
<td>1/4</td>
<td>1/2</td>
<td>1/8</td>
</tr>
</tbody>
</table>

Here $F_4$ is any forest having 4 edges, and $K_4^-$ is the diamond graph. Note that all rows sum to one, so there is no need to have more columns. The spectrum we’re looking for must satisfy $\lambda_0 = 1$, and so $c_0 = 1$. We can also deduce other constraints. In order to get a bound of 1/8, the spectrum must satisfy $\lambda_{\text{min}} = -1/7$, and this gives us several inequality constraints. Furthermore, if we plug in a triangle-junta into Hoffman’s bound then all the inequalities must be tight. That means that for every non-zero Fourier coefficient in this family, the corresponding eigenvalue must be $\lambda_{\text{min}}$. This gives us more constraints. In this way, we can deduce $c_1 = -5/7$, $c_2 = -1/7$ and $4c_3 + c_4 = 3/7$. This gives us one degree of freedom. We arbitrarily choose $c_4 = 0$ to obtain the simplest possible expression,

$$(-1)^{|H|} \left( \frac{1}{7}q_0(H) - \frac{5}{7}q_1(H) - \frac{1}{7}q_2(H) + \frac{3}{28}q_3(H) \right).$$

A miracle happens and the minimal value of this expression, over all graphs, is $-1/7$. Intuitively, for large $|H|$ this expression is close to zero, while for small $|H|$ we engineered it to obtain the correct eigenvalues. This leaves open the case of medium $|H|$, which must be tediously checked. We conclude that an odd-cycle-agreeing family has measure at most 1/8, proving the Simonovits–Sós conjecture.
Simonovits and Sós conjectured that triangle-juntas are the unique maximal families. In order to prove this, we need to fudge a bit with our spectrum. The problem is that while \( \lambda_{\text{min}} = -1/7 \), this is obtained on two many eigenvalues: on those corresponding to forests of 1, 2, 4 edges, triangles and diamonds. Fortunately, we can fix that. Consider the expression

\[
(-1)^{|H|} \left( \frac{1}{7} q_0(H) - \frac{5}{7} q_1(H) - \frac{1}{7} q_2(H) + \frac{3}{28} q_3(H) + \frac{2}{119} \sum_F q_F(H) - \frac{2}{119} q_{\Box}(H) \right),
\]

where the sum ranges over all forests having exactly 4 edges. Some calculation shows that the minimal eigenvalue is now attained only on forests of 1 or 2 edges and triangles, and furthermore all other eigenvalues are at least \(-135/952 > -1/7\). Suppose now that we have an odd-cycle-agreeing family of measure \( 1/8 \). All the inequalities in Hoffman’s bound must be tight, and so its Fourier expansion is supported on sets of at most 3 edges. Some simple arguments show that the family must depend on at most 3 edges, and so must be a triangle-semijunta (all graphs which intersect a fixed triangle in a specific way).

One advantage of the spectral approach is that it implies more than just an upper bound and a description of the optimal families: we can also get a stability result, showing that nearly-optimal families are close to optimal families. Consider an odd-cycle-agreeing family \( \mathcal{F} \) of measure \( 1/8 - \epsilon \). Since there is a gap between the minimal eigenvalue and all other ones, an analysis of Hoffman’s bound shows that a \( 1 - O(\epsilon) \) fraction of the Fourier expansion of the characteristic function \( f \) of the family lies on the first \( 3 + 1 \) levels. A deep theorem of Kindler and Safra [4] then shows that \( \mathcal{F} \) is \( O(\epsilon) \)-close to a family \( \mathcal{G} \) depending on \( O(1) \) coordinates. If the family \( \mathcal{G} \) is not odd-cycle-agreeing then consider two non-odd-cycle-agreeing graphs \( G, H \in \mathcal{G} \); we can assume that \( G, H \) are supported on the \( O(1) \) coordinates. For each graph \( K \) on the complement of these coordinates, \( \mathcal{F} \) can contain at most one of \( G \cup K \) and \( H \cup \overline{K} \); therefore \( \mathcal{F} \) is \( \Omega(1) \)-far from \( \mathcal{G} \), and by assuming that \( \epsilon \) is small enough, we can rule out this case. There are finitely many odd-cycle-agreeing families which are not triangle-semijuntas, and so if \( \epsilon \) is small enough, we can also rule out \( \mathcal{G} \) being one of them. We conclude that for \( \epsilon \) small enough, \( \mathcal{F} \) is \( O(\epsilon) \)-close to a triangle-semijunta. We can drop the assumption on \( \epsilon \) by adjusting the big \( O \) constant, obtaining the complete stability result.

References


