Friedgut–Kalai–Naor for $S_n$

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Suppose $f : \{\pm 1\}^n \to \{\pm 1\}$ is linear:

$$f(x_1, \ldots, x_n) = c_0 + \sum_i c_i x_i.$$ 

Then $f \in \{\pm 1, \pm x_i\}$.

Application: uniqueness for $\mu_p$-EKR.
Suppose $f : \{\pm 1\}^n \to \{\pm 1\}$ is almost linear:

$$f(x_1, \ldots, x_n) \approx c_0 + \sum_i c_i x_i.$$ 

Then $f \approx \{\pm 1, \pm x_i\}$.

Application: stability for $\mu_p$-EKR.
Suppose $f : \{\pm 1\}^n \to \{\pm 1\}$ is almost linear:

$$\mathbb{E}[(f(x_1, \ldots, x_n) - (c_0 + \sum_i c_ix_i))^2] = \epsilon.$$ 

Then $\mathbb{E}[(f - g)^2] = O(\epsilon)$ for some $g \in \{\pm 1, \pm x_i\}$.

Application: stability for $\mu_p$-EKR.
Suppose $f: S_n \rightarrow \{0, 1\}$ is linear:

$$f = \sum_{i,j} c_{ij} T_{ij}$$

where $T_{ij}$ is the set of permutations sending $i$ to $j$. Then $f$ is a disjoint union of $T_{ij}$’s, i.e. a dictator: $f(\pi)$ depends only on $\pi(i)$ or $\pi^{-1}(j)$.

Application: uniqueness for $S_n$-EKR.
Conjecture

Suppose $f : S_n \to \{0, 1\}$ is almost linear:

$$f \approx \sum_{i,j} c_{ij} T_{ij}.$$

Then $f \approx$ union of $T_{ij}$’s.

Application: stability for $S_n$-EKR.
Conjecture

Suppose \( f : S_n \rightarrow \{0, 1\} \) is almost linear:

\[
E[(f - \sum_{i,j} c_{ij} T_{ij}))^2] = \epsilon.
\]

Then \( E[(f - g)^2] = O(\epsilon) \) for some \( g = \text{union of } T_{ij}'s. \)

Application: stability for \( S_n\)-EKR.
Suppose $f : S_n \to \{0, 1\}$ is almost linear:

$$\mathbb{E}[(f - \sum_{i,j} c_{ij} T_{ij})^2] = \epsilon \mathbb{E}[f].$$

Then for some $g = \text{union of } c = \lceil n \mathbb{E}[f] \rceil \ T_{ij}$'s,

$$\mathbb{E}[(f - g)^2] = O(\sqrt{\epsilon} + \frac{1}{n})c \mathbb{E}[f].$$

Cannot guarantee cosets to be disjoint!
(Think of $T_{11} + T_{22} - T_{12,12}$.)
Suppose $f : S_n \rightarrow \{0, 1\}$ is almost linear:

$$\mathbb{E}[(f - \sum_{i,j} c_{ij} T_{ij})^2] = \epsilon.$$

Then for some dictator $g = \text{disjoint union of } \lceil n \mathbb{E}[f] \rceil T_{ij}$'s,

$$\mathbb{E}[(f - g)^2] = O\left(\frac{1}{\eta} (\epsilon^{1/7} + \frac{1}{n^{1/3}})\right),$$

where $\eta = \min(\mathbb{E} f, 1 - \mathbb{E} f)$. 
EFF2 — Setup

- $f$ is a $\pm 1$ function.
- $f_1 =$ projection of $f$ into span of $T_{ij}$'s.
- $\mathbb{E}[(f - f_1)^2] = \epsilon$.
- For simplicity, assume $\mathbb{E} f = 0$.

Canonical representation of $f_1$:

$$f_1 = \sum_{i,j} a_{ij} T_{ij}, \quad \text{where } a_{ij} = (n - 1) \langle f, T_{ij} \rangle,$$

$$f_1(\pi) = \sum_i a_{i\pi(i)}.$$
Matrix representation

Matrix \((a_{ij})\) for \(f = T_{11} \cup \cdots \cup T_{1(n/2)}:\)

\[
\begin{pmatrix}
1 - \frac{1}{n} & \cdots & 1 - \frac{1}{n} & \frac{1}{n} - 1 & \cdots & \frac{1}{n} - 1 \\
-\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\
-\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}
\end{pmatrix}
\]

\(f_1(\pi) = \sum_i a_{i\pi(i)}\) (generalized diagonal)
Step 1

Choose $X, Y \subseteq [n]$ randomly of same size.

\[ g \triangleq f_1|_{\pi: \pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}}), \]

\[ g_1(\pi_1) = \sum_{i \in X} a_{i\pi_1(i)}, \quad g_2(\pi_2) = \sum_{i \in \overline{X}} a_{i\pi_2(i)}. \]

Whp over choice of $X, Y$:

- $g$ is close to $\pm 1$
- $\mathbb{E} g_1, \mathbb{E} g_2$ close to 0

($X, Y$ are reasonable.)
Step 1 (cont.)

Whp over choice of $X, Y$:
- $g$ is close to $±1$
- $\mathbb{E} g_1, \mathbb{E} g_2$ close to 0

Inputs to $g_1, g_2$ are independent.

How can $g = g_1 + g_2$ be close to $±1$?
- One of $g_1, g_2$ must be almost constant $−X$, other close to $X ± 1$.
- Since $\mathbb{E} g_1 ≈ \mathbb{E} g_2 ≈ 0$, $X ≈ 0$.
- So one of $g_1, g_2$ is almost zero, other close to $±1$. 
Step 2

Whp over $\pi \in S_n$:

- Most $X, \pi(X)$ are reasonable.
- So for most $X$,

$$\sum_{i \in X} a_{i\pi(i)} \approx 0, \quad \sum_{i \in \overline{X}} a_{i\pi(i)} \approx \pm 1$$

or vice versa.

- This can only happen if one $a_{i\pi(i)}$ is large $(\approx \pm 1)$ and all other $a_{i\pi(i)}$ are small $(\approx 0)$.

Conclusion: almost all generalized diagonals in $(a_{ij})$ have exactly one large entry.
Step 3

If all generalized diagonals in \((a_{ij})\) have one large entry then
- All entries in some row or column are large.

We prove a stability version of this result:
- In some row/column, almost all entries are large.
- Every row/column sums to zero, so approximately half are \(+1\), half are \(-1\).
- Union of \(+1\) cosets approximates \(f\).
EFF1 — Setup

- $f$ is a $\{0, 1\}$ function.
- $f_1$ = projection of $f$ into span of $T_{ij}$'s.
- $\mathbb{E}[(f - f_1)^2] = \epsilon$.
- For simplicity, assume $\mathbb{E} f = 1/n$.

Canonical representation of $f_1$:

$$h = \sum_{i,j} b_{ij} T_{ij}, \quad \text{where } b_{ij} = n \langle f, T_{ij} \rangle - \frac{1}{n}.$$  

$$h = \frac{n}{n-1} f_1 - \frac{1}{n-1} \approx f_1.$$  

Note $\mathbb{E} h = 0$, $h \geq -1/(n-1)$.  

Matrix representation

Matrix \((b_{ij})\) for \(f = T_{11}\):

\[
\begin{pmatrix}
1 - \frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\
-\frac{1}{n} & \frac{1}{n(n-1)} & \cdots & \frac{1}{n(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n} & \frac{1}{n(n-1)} & \cdots & \frac{1}{n(n-1)}
\end{pmatrix}
\]

- Rows and columns sum to 0.
- \(\sum_{i,j} b_{ij}^2 \approx \sum_{i,j} b_{ij}^3 \approx 1.\)
Moments of the matrix

Moments:

\[ \mathbb{E}[h^2] = \frac{1}{n-1} \sum_{i,j} b_{ij}^2, \]
\[ \mathbb{E}[h^3] = \frac{n}{(n-1)(n-2)} \sum_{i,j} b_{ij}^3. \]

Estimating the moments:

- \( f \approx f_1 \implies \mathbb{E}[h^2] \approx \frac{1}{n}. \)
- \( \mathbb{E}[h^3] \geq \frac{1}{n} \) follows from:
  1. \( \mathbb{E} h = 0, \)
  2. \( h + 1 \geq 0, \)
  3. \( h + 1 \) is \( L_2 \)-close to \( f + 1, \)
  4. \( f + 1 \) is 1 w.p. \( 1 - 1/n \) and 2 w.p. \( 1/n. \)
Finishing the proof

- \( \mathbb{E}[h^2] \approx \frac{1}{n} \) and \( \mathbb{E}[h^3] \approx \frac{1}{n} \).
- So \( \sum_{i,j} b_{ij}^2 \approx 1 \) and \( \sum_{i,j} b_{ij}^3 \approx 1 \).
- So \( \sum_{i,j} b_{ij}^2(1 - b_{ij}) \approx 0 \).
- So \( b_{ij} \)'s are close to \( \{0, 1\} \).
- Since \( \sum_{i,j} b_{ij}^2 \approx 1 \), exactly one \( b_{ij} \) is close to 1.
- So \( f \approx T_{ij} \).
Suppose $f: \binom{[n]}{k} \to \{-1, 1\}$ is linear:

$$f(x_1, \ldots, x_n) = \sum_i c_i x_i.$$  

Then $f \in \{ \pm 1, \pm x_i \}$.

Application: uniqueness for EKR.
Suppose $f : \binom{[n]}{k} \to \{\pm 1\}$ is almost linear:

$$f(x_1, \ldots, x_n) \approx \sum_i c_i x_i.$$ 

Then $f \approx \{\pm 1, \pm x_i\}$.

Application: stability for EKR.
Suppose \( f : \begin{pmatrix} n \end{pmatrix} \to \{ \pm 1 \} \) is almost linear:

\[
\mathbb{E}[(f(x_1, \ldots, x_n) - (\sum_i c_i x_i))^2] = \epsilon \leq c \frac{k^2}{n^2}.
\]

Then \( \mathbb{E}[(f - g)^2] = O(\epsilon) \) for some \( g \in \{ \pm 1, \pm x_i \} \).

Application: stability for EKR.