Structure theorems for almost low degree functions

Yuval Filmus

Institute for Advanced Study

April 8, 2014
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   - Connection to structure theorems
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Section 1

Theorems
Suppose $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is linear:

$$f(x_1, \ldots, x_n) = c_0 + \sum_{i=1}^{n} c_i x_i.$$
Suppose \( f: \{-1,1\}^n \rightarrow \{-1,1\} \) is linear:

\[
f(x_1, \ldots, x_n) = c_0 + \sum_{i=1}^{n} c_i x_i.
\]

**Theorem:** \( f \) depends on at most one coordinate.
Almost linear functions

Suppose $f: \{-1, 1\}^n \to \{-1, 1\}$ is almost linear:

$$\mathbb{E} \left[ \left( c_0 + \sum_{i=1}^{n} c_i x_i - f \right)^2 \right] = \epsilon.$$

Distribution over $\{-1, 1\}^n$: uniform or $\mu_p$.

$$\mu_p(x_1, \ldots, x_n) = p^{\#\{i: x_i = -1\}} (1 - p)^{\#\{i: x_i = 1\}}.$$
Almost linear functions

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Distribution over $\{-1, 1\}^n$: uniform or $\mu_p$.

$$\mu_p(x_1, \ldots, x_n) = p^{|\{i: x_i = -1\}|} (1 - p)^{|\{i: x_i = 1\}|}.$$ 

Theorem: $f$ is $O(\epsilon)$-close to a linear Boolean function. 

*(Friedgut–Kalai–Naor, 2002)*
If \( f : \{-1, 1\}^n \to \{-1, 1\} \) has degree \( d \) then
\( f \) depends on \( \leq d2^d \) variables.

\((\text{Nisan–Szegedy, 1994})\)
If $f : \{-1, 1\}^n \to \{-1, 1\}$ has degree $d$ then $f$ depends on $\leq d2^d$ variables.

(Nisan–Szegedy, 1994)

If $f : \{-1, 1\}^n \to \{-1, 1\}$ is $\epsilon$-close to a function of degree $d$ then $f$ is $O(\epsilon)$-close to a Boolean function of degree $d$.

(Kindler–Safra, 2002)
The slice is \( \binom{[n]}{k} \).

Usually assume \( \delta \leq \frac{k}{n} \leq 1 - \delta \).

Can identify the slice with

\[
\left\{ (x_1, \ldots, x_n) \in \{\pm 1\}^n : \sum_{i=1}^n x_i = 2k - n \right\}.
\]
Suppose $f : \binom{[n]}{k} \rightarrow \{-1, 1\}$ is linear:

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i.$$ 

Theorem: $f$ depends on at most one coordinate.
Almost linear functions

Suppose $f: \binom{[n]}{k} \to \{-1, 1\}$ is almost linear:

$$\mathbb{E} \left[ \left( \sum_{i=1}^{n} c_i x_i - f \right)^2 \right] = \epsilon.$$

Uniform distribution on $\binom{[n]}{k}$.

Theorem: $f$ is $O(\epsilon)$-close to a linear Boolean function.

(F. et al., 2013+)
If \( f : \binom{[n]}{k} \rightarrow \{-1, 1\} \) has degree \( d \) then \( f \) depends on \( \exp(d) \) variables.

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If \( f : \binom{[n]}{k} \to \{-1, 1\} \) has degree \( d \) then
\( f \) depends on \( \exp(d) \) variables.

\((F. \ et \ al., \ 2013+)\)

Conjecture: If \( f : \binom{[n]}{k} \to \{-1, 1\} \) is \( \epsilon \)-close to a function of degree \( d \) then \( f \) is \( O(\epsilon) \)-close to a \textit{Boolean} function of degree \( d \).
The symmetric group is $S_n$.

Can identify $S_n$ with permutation matrices $X = (x_{ij})_{i,j=1}^n$. Each entry is 0/1, each row and each column sums to 1.
Suppose \( f : S_n \to \{0, 1\} \) is linear:

\[
f(X) = \sum_{i,j=1}^{n} c_{ij} X_{ij}.
\]

Theorem: \( f \) depends on at most one row or one column.

\[
f(\pi) = \llbracket \pi(i) \in J \rrbracket
\]

or

\[
f(\pi) = \llbracket \pi^{-1}(j) \in I \rrbracket
\]

*(Ellis, Friedgut and Pilpel, 2011)*
Almost linear functions

Suppose \( f : S_n \rightarrow \{0, 1\} \) is almost linear:

\[
\mathbb{E} \left[ \left( \sum_{i,j=1}^{n} c_{ij} x_i - f \right)^2 \right] = \epsilon.
\]

Uniform distribution on \( S_n \).

Theorem: If \( f \) is balanced, \( f \) is \( O(\epsilon^{1/7}) \)-close to a linear Boolean function.

Theorem: If \( f \) is sparse, \( f \) is \( O(\epsilon^{1/2}) \)-close to a function of the form

\[
\max(x_{i_1j_1}, \ldots, x_{i_rj_r}),
\]

i.e., characteristic function of a union of double cosets

\[
T_{ij} = \{ \pi \in S_n : \pi(i) = j \}.
\]

Sparse means \( \mathbb{E}[f] = c/n \) for small \( c \).

(Ellis, F., Friedgut, 2014)
If $f : S_n \rightarrow \{0, 1\}$ has degree $d$ then $f$ can be written as a sum of disjoint monomials of degree $d$. I.e., $f$ is characteristic function of disjoint sum of double $d$-cosets $T_{i_1j_1} \cap \cdots \cap T_{i_dj_d}$.

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\( f \) can be written as a sum of disjoint monomials of degree \( d \).

I.e., \( f \) is characteristic function of disjoint sum of double \( d \)-cosets

\[ T_{i_1 j_1} \cap \cdots \cap T_{i_d j_d}. \]

*(Ellis, Friedgut and Pilpel, 2011)*

Theorem: If \( f \) is sparse, \( f \) is \( O(\epsilon^{1/2}) \)-close to the characteristic function of a union of double \( d \)-cosets.

Sparse means \( \mathbb{E}[f] = c/n^d \) for small \( c \).

*(Ellis, F., Friedgut, 2014)*
Section 2

Applications
If $k < n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$ 

Equality only for $\mathcal{F} = \{S \in \binom{[n]}{k} : i \in S\}$ ("star").
Erdős–Ko–Rado theorems

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If $p < 1/2$ and $\mathcal{F} \subseteq 2^{[n]}$ is intersecting, then $\mu_p(\mathcal{F}) \leq p$.
Equality only for $\mathcal{F} = \{ S \in 2^{[n]} : i \in S \}$.

*(Friedgut, 2008)*
Erdős–Ko–Rado theorems

Erdős–Ko–Rado theorem (1938/1961): If $k < n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then

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(Friedgut, 2008)

If $\mathcal{F} \subseteq S_n$ is intersecting, then $|\mathcal{F}| \leq (n-1)!$. Equality only for $\mathcal{F} = \{ \pi \in S_n : \pi(i) = j \}$.

(Deza–Frankl, 1977); (Cameron–Ku, 2003)
If $k < n/2$, $\mathcal{F} \subseteq {\binom{[n]}{k}}$ is intersecting, and $|\mathcal{F}| \approx \binom{n-1}{k-1}$, then $\mathcal{F}$ is close to a star.

(Frankl, 1987)

If $p < 1/2$, $\mathcal{F} \in 2^{[n]}$ is intersecting, and $\mu_p(\mathcal{F}) \approx p$, then $\mathcal{F}$ is close to a star.

(Friedgut, 2008)

If $\mathcal{F} \subseteq S_n$ is intersecting and $|\mathcal{F}| \approx (n-1)!$, then $\mathcal{F}$ is close to a star.

(Ellis, 2009)
Stability and structure theorems

Stability theorems follow from structure theorems. Example: Intersecting families in $2^{[n]}$.

Let $f$ be characteristic function of an intersecting family. Friedgut constructs a quadratic form $Q$ such that $\langle f, Qf \rangle = 0$.

Spectral decomposition of $Q$ implies

$$\sum_{S \subseteq [n]} \left( -\frac{p}{1-p} \right)^{|S|} \hat{f}(S)^2 = 0.$$ 

Also know $\hat{f}(\emptyset) = \sum_S \hat{f}(S)^2 = \mu_p(\mathcal{F})$. 
Stability and structure theorems

For characteristic function $f$ of intersecting family $\mathcal{F}$:

$$\sum_{S \subseteq [n]} \left( -\frac{p}{1-p} \right)^{|S|} \hat{f}(S)^2 = 0.$$ 

Also know $\hat{f}(\emptyset) = \sum_S \hat{f}(S)^2 = \mu_p(\mathcal{F})$.

- $\mu_p(\mathcal{F}) \leq p$.
- If $\mu_p(\mathcal{F}) = p$ then $\hat{f}$ is supported on first two levels.
- If $\mu_p(\mathcal{F}) \approx p$ then $\hat{f}$ is concentrated on first two levels.

Friedgut–Kalai–Naor gives stability.
Multiple intersections

If \( n \geq (t + 1)(k - t - 1) \) and \( \mathcal{F} \) is \( t \)-intersecting, then

\[
|\mathcal{F}| \leq \binom{n - t}{k - t}.
\]

Equality only for \( \mathcal{F} = \{ S \in \binom{[n]}{k} : i_1, \ldots, i_t \in S \} \) ("\( t \)-star").

\((Frankl, 1984)\)
Multiple intersections

If $n \geq (t + 1)(k - t - 1)$ and $\mathcal{F}$ is $t$-intersecting, then

$$|\mathcal{F}| \leq \binom{n - t}{k - t}.$$  

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*(Frankl, 1984)*

If $p < 1/(t + 1)$ and $\mathcal{F} \subseteq 2^{[n]}$ is $t$-intersecting, then $\mu_p(\mathcal{F}) \leq p^t$.

Equality only for $\mathcal{F} = \{ S \in 2^{[n]} : i_1, \ldots, i_t \in S \}$.

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Multiple intersections

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Equality only for \( \mathcal{F} = \{ S \in 2^{[n]} : i_1, \ldots, i_t \in S \} \).

(Friedgut, 2008)

If \( n \geq C_t \) and \( \mathcal{F} \subseteq S_n \) is \( t \)-intersecting, then \( |\mathcal{F}| \leq (n - t)! \).

Equality only for \( \mathcal{F} = \{ \pi \in S_n : \pi(i_1) = j_1, \ldots, \pi(i_t) = j_t \} \).

\( C_t \) should be equal to \( 2t + 1 \).

(Ellis, Friedgut and Pilpel, 2011)
Conjecture: if \( n \geq (t + 1)(k - t - 1) \), \( \mathcal{F} \) is \( t \)-intersecting, and \( |\mathcal{F}| \approx \binom{n-t}{k-t} \), then \( \mathcal{F} \) is close to a \( t \)-star.

If \( p < 1/(t + 1) \), \( \mathcal{F} \in 2^n \) is \( t \)-intersecting, and \( \mu_p(\mathcal{F}) \approx p^t \), then \( \mathcal{F} \) is close to a \( t \)-star.

\textit{(Friedgut, 2008), proof uses (Kindler–Safra, 2002)}

If \( n \geq C_t \), \( \mathcal{F} \subseteq S_n \) is \( t \)-intersecting, and \( |\mathcal{F}| \approx (n - t)! \), then \( \mathcal{F} \) is close to a \( t \)-star.

\textit{(Ellis, 2011)}
Section 3

Proofs
Proof sketch

Suppose \( f : \binom{[n]}{n/2} \to \{-1, 1\} \) is \( \epsilon \)-close to a linear function:

\[
f(x_1, \ldots, x_n) \approx \sum_{i=1}^{n} c_i x_i =: \ell.
\]

(Recall \( \sum_{i=1}^{n} x_i = 0 \).)
Proof sketch

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(Recall \( \sum_{i=1}^{n} x_i = 0 \).)

1. For each \( i \), either \( x_i \approx \pm 1 \) or \( x_i \approx 0 \).
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1. For each \( i \), either \( x_i \approx \pm 1 \) or \( x_i \approx 0 \).
2. \( x_i \approx \pm 1 \) for at most one \( i \).
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1. For each $i$, either $x_i \approx \pm 1$ or $x_i \approx 0$.
2. $x_i \approx \pm 1$ for at most one $i$.
3. Reduce to the case $x_i \approx 0$ for all $i$. 

Apply Friedgut–Kalai–Naor.
Proof sketch

Suppose \( f : \binom{[n]}{n/2} \to \{-1, 1\} \) is \( \epsilon \)-close to a linear function:

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Applying Friedgut–Kalai–Naor

A subcube is a subset of the slice of the form

\[ \{a_1, b_1\} \times \cdots \times \{a_{n/2}, b_{n/2}\}. \]

Corresponding restriction of \( \ell \) is

\[
g(y_1, \ldots, y_n) = C + \frac{1}{2} \sum_{i=1}^{n/2} (c_{a_i} - c_{b_i}) y_i. \]
Applying Friedgut–Kalai–Naor

A *subcube* is a subset of the slice of the form

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Friedgut–Kalai–Naor over a random subcube implies

\[
\frac{n}{2} \sum_{i,j=1}^{n} (c_i - c_j)^2 = O(\epsilon).
\]
Applying Friedgut–Kalai–Naor

A subcube is a subset of the slice of the form

\[ \{a_1, b_1\} \times \cdots \times \{a_{n/2}, b_{n/2}\} \].

Corresponding restriction of \( \ell \) is

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Friedgut–Kalai–Naor over a random subcube implies

\[ \frac{n}{2} \sum_{i,j=1}^{n} (c_i - c_j)^2 = O(\epsilon). \]

Left-hand side upper bounds the variance of \( \ell \).

Since \( \nabla[\ell] = O(\epsilon) \), \( \mathbb{E}[\ell] \approx \pm 1 \). We are done since \( f \approx \ell \).
Proof sketch

Suppose \( f : S_n \to \{0, 1\} \) is sparse (\( \mathbb{E} f = c/n \)) and close to its linear projection \( \mathcal{L} \).
Proof sketch

Suppose $f : S_n \to \{0, 1\}$ is sparse ($\mathbb{E} f = c/n$) and close to its linear projection $\ell$.

1. Let $b_{ij} = |F \cap T_{ij}|/|T_{ij}| - |F|/|S_n|$.

2. Let $h = \sum_{ij} b_{ij} x_{ij}$.

3. $\mathbb{E}[h^2] \approx 1/n \sum_{ij} b_{ij}^2$.

4. $\mathbb{E}[h^3] \approx 1/n \sum_{ij} b_{ij}^3$.

5. Since $f \approx \ell$, can estimate $\mathbb{E}[h^2] \approx c/n$.

6. Since $h \geq 0$ and $h$ is close to Boolean, $\mathbb{E}[h^3] \geq c/n$.

7. So $\sum_{ij} b_{ij}^2 (1 - b_{ij}) \lesssim 0$.

8. So for each $i, j$, either $b_{ij} \approx 0$ or $b_{ij} \approx 1$.

9. Roughly $c$ of the $b_{ij}$ are close to 1.

F is approximated by union of the corresponding $T_{ij}$. 

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Proof sketch

Suppose $f : S_n \to \{0, 1\}$ is sparse ($\mathbb{E}f = c/n$) and close to its linear projection $\ell$.

1. Let $b_{ij} = \frac{|\mathcal{F} \cap T_{ij}|}{|T_{ij}|} - \frac{|\mathcal{F}|}{|S_n|}$.
2. Let $h = \sum_{ij} b_{ij} x_{ij}$.
Proof sketch

Suppose \( f : S_n \rightarrow \{0, 1\} \) is sparse (\( \mathbb{E}f = c/n \)) and close to its linear projection \( \ell \).

1. Let \( b_{ij} = |\mathcal{F} \cap T_{ij}| / |T_{ij}| - |\mathcal{F}| / |S_n| \).
2. Let \( h = \sum_{ij} b_{ij} x_{ij} \).
3. \( \mathbb{E}[h^2] \approx \frac{1}{n} \sum_{ij} b_{ij}^2 \), \( \mathbb{E}[h^3] \approx \frac{1}{n} \sum_{ij} b_{ij}^3 \).
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4. Since $f \approx \ell$, can estimate $\mathbb{E}[h^2] \approx c/n$.
5. Since $h \gtrsim 0$ and $h$ is close to Boolean, $\mathbb{E}[h^3] \gtrsim c/n$. 
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Suppose \( f : S_n \rightarrow \{0,1\} \) is sparse (\( \mathbb{E}f = c/n \)) and close to its linear projection \( \ell \).

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4. Since \( f \approx \ell \), can estimate \( \mathbb{E}[h^2] \approx c/n \).
5. Since \( h \gtrsim 0 \) and \( h \) is close to Boolean, \( \mathbb{E}[h^3] \gtrsim c/n \).
6. So \( \sum_{ij} b_{ij}^2(1 - b_{ij}) \lesssim 0 \).
Proof sketch

Suppose $f : S_n \to \{0, 1\}$ is sparse ($\mathbb{E} f = c/n$) and close to its linear projection $\ell$.

1. Let $b_{ij} = |\mathcal{F} \cap T_{ij}| / |T_{ij}| - |\mathcal{F}| / |S_n|$.

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8. Roughly \( c \) of the \( b_{ij} \) are close to 1.
Proof sketch

Suppose $f : S_n \to \{0, 1\}$ is sparse ($\mathbb{E}f = c/n$) and close to its linear projection $\ell$.

1. Let $b_{ij} = |\mathcal{F} \cap T_{ij}| / |T_{ij}| - |\mathcal{F}| / |S_n|$. 
2. Let $h = \sum_{ij} b_{ij} x_{ij}$. 
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4. Since $f \approx \ell$, can estimate $\mathbb{E}[h^2] \approx c/n$. 
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7. So for each $i, j$, either $b_{ij} \approx 0$ or $b_{ij} \approx 1$. 
8. Roughly $c$ of the $b_{ij}$ are close to 1. 
9. $\mathcal{F}$ is approximated by union of the corresponding $T_{ij}$. 

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Proof sketch

Suppose $f : S_n \rightarrow \{-1, 1\}$ is balanced ($\mathbb{E}f = 0$) and close to its linear projection $\ell$.
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Suppose $f : S_n \to \{-1, 1\}$ is balanced ($\mathbb{E}f = 0$) and close to its linear projection $\ell$.

1. Let $a_{ij} = \frac{n-1}{n!} |\mathcal{F} \cap T_{ij}|$, so $f(\pi) = \sum_{i=1}^{n} a_{i\pi(i)}$. 

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Suppose $f : S_n \to \{-1, 1\}$ is balanced ($\mathbb{E}f = 0$) and close to its linear projection $\ell$.

1. Let $a_{ij} = \frac{n-1}{n!} |\mathcal{F} \cap T_{ij}|$, so $f(\pi) = \sum_{i=1}^{n} a_{i\pi(i)}$.
2. For random $X, Y$, consider $f|_{\pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}})$. 

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Suppose $f : S_n \to \{-1, 1\}$ is balanced ($\mathbb{E} f = 0$) and close to its linear projection $\ell$.

1. Let $a_{ij} = \frac{n-1}{n!} |\mathcal{F} \cap T_{ij}|$, so $f(\pi) = \sum_{i=1}^{n} a_{i\pi(i)}$.
2. For random $X, Y$, consider $f|_{\pi(X)=Y} = g_1(\pi|X) + g_2(\pi|\bar{X})$.
3. For most $X, Y$, $f|_{\pi(X)=Y}$ close to Boolean ("$(X, Y)$ good").
Proof sketch

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4. Since \( g_1, g_2 \) are "independent", one must be \( \approx 0 \), the other close to Boolean.
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5. For most $\pi \in S_n$, most pairs $(X, \pi(X))$ are good.
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6. So \( g_1(\pi|X) = \sum_{i \in X} a_{i\pi(i)} \approx 0, 1 \) (or 0, \(-1\)) w.p. \( \approx 1 \).
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7. Friedgut–Kalai–Naor: $a_{i\pi(i)} \approx 0$ for all $i$ with $\leq 1$ exception.
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8. Can only happen if all "strong" entries of $(a_{ij})$ concentrated on one row or column.
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1. Let $a_{ij} = \frac{n-1}{n!} |\mathcal{F} \cap T_{ij}|$, so $f(\pi) = \sum_{i=1}^{n} a_{i\pi(i)}$.
2. For random $X, Y$, consider $f|_{\pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}})$.
3. For most $X, Y$, $f|_{\pi(X)=Y}$ close to Boolean ("$(X, Y)$ good").
4. Since $g_1, g_2$ are "independent", one must be $\approx 0$, the other close to Boolean.
5. For most $\pi \in S_n$, most pairs $(X, \pi(X))$ are good.
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8. Can only happen if all "strong" entries of $(a_{ij})$ concentrated on one row or column.
9. $f$ essentially depends only on this strong row or column.