Boolean function analysis: beyond the hypercube

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What is Boolean function analysis?

Dimension-independent properties of functions $\{0,1\}^n \to \{0,1\}$

Many applications to combinatorics and computational complexity

Motivating example: Erdős–Ko–Rado theorem

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3. $|\mathcal{F}| \approx \binom{n-1}{k-1} \implies \mathcal{F} \approx$ a star.

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   Boolean *almost* degree 1 function is *almost* a dictator.

Classical Boolean function analysis

Fundamental theorem

Every function $f\colon \{\pm 1\}^n \to \mathbb{R}$ has unique expansion as multilinear polynomial, the *Fourier expansion*:

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \hat{f}(S) x_S, \quad \text{where } x_S = \prod_{i \in S} x_i.$$

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Dictator: function depending on one coordinate.
d-Junta: function depending on d coordinates.
$\deg f \leq d$ iff f is linear combination of d-juntas.

Boolean degree 1 functions

Question

Suppose $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has degree 1.

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If $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has degree 1 then

$$f \in \{\pm 1, \pm x_1, \dots, \pm x_n\}.$$

Boolean almost degree 1 functions

Refined question

Suppose $f\colon \{\pm 1\}^n \to \{\pm 1\}$ satisfies

$$\mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n}[(f(x) - g(x))^2] = \epsilon$$

for some $g\colon \{\pm 1\}^n \to \mathbb{R}$ of degree 1.

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Friedgut–Kalai–Naor (FKN) theorem

Suppose $f\colon \{\pm 1\}^n \to \{\pm 1\}$ satisfies $\|f^{>1}\|^2 = \epsilon$. Then

$$\Pr[f \neq h] = O(\epsilon) \text{ for some } h \in \{\pm 1, \pm x_1, \ldots, \pm x_n\}.$$

Boolean function analysis on the slice

The *slice* or *Johnson scheme* is

$$\binom{[n]}{k} = \left\{ (x_1, \ldots, x_n) \in \{0,1\}^n : \sum_{i=1}^{n} x_i = k \right\}.$$

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Fundamental theorem (Dunkl)

Every function $f\colon \binom{[n]}{k} \to \mathbb{R}$ has unique expansion as multilinear polynomial P of degree $\leq \min(k, n-k)$ such that

$$\sum_{i=1}^{n} \frac{\partial P}{\partial x_i} = 0.$$

Examples: $1, (x_1 - x_2), (x_1 - x_2)(x_3 - x_4), \dots$

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Dictator theorem holds (except for trivial cases).
FKN theorem holds for $0 \ll k/n \ll 1$.

Erdős–Ko–Rado theorem

Spectral argument of Lovász

Let $k = pn$, $p < 1/2$.
If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $\mathcal{F}$ is not too small then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} \left(1 - C\|1_{\mathcal{F}}^{\geq 1}\|^2\right).$$

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Corollaries

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Corollaries

1. $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
2. $|\mathcal{F}| = \binom{n-1}{k-1} \implies \deg 1_{\mathcal{F}} = 1$.
   Dictator theorem: $\mathcal{F}$ is a star.
3. $|\mathcal{F}| = (1-\epsilon)\binom{n-1}{k-1} \implies \|1_{\mathcal{F}}^{\geq 1}\|^2 = O(\epsilon)$.
   FKN theorem: $\mathcal{F}$ is $O(\epsilon)$-close to a star.

FKN theorem for small k?

Let $p := k/n = o(1)$ and $\epsilon \gg p^2$.
Consider $g\colon \binom{[n]}{k} \to \mathbb{R}$ defined as

$$g := x_1 + \cdots + x_{\sqrt{\epsilon}/p}$$

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This shows that

- $\Pr[g = 0] \approx 1 - \sqrt{\epsilon}$.
- $\Pr[g = 1] \approx \sqrt{\epsilon} - \epsilon$.
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Therefore

$$g \overset{O(\epsilon)}{\approx} f := x_1 \vee \cdots \vee x_{\sqrt{\epsilon}/p}$$

FKN theorem for small k

FKN theorem on the slice (F.)

Let $p := k/n \leq 1/2$.
If $f\colon \binom{[n]}{k} \to \{0,1\}$ satisfies $\|f^{>1}\|^2 = \epsilon$ then either f or $1-f$ is $O(\epsilon)$-close to a disjunction of m variables, where

$$m = \max\left\{1, O\left(\frac{\sqrt{\epsilon}}{p}\right)\right\}.$$

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Dictator theorem on the slice

If $f\colon \binom{[n]}{k} \to \{0,1\}$ has degree 1 and $k \neq 1, n-1$ then

$$f \in \{0, 1, x_1, 1-x_1, \ldots, x_n, 1-x_n\}.$$

Symmetric group

The symmetric group is

$$S_n = \{\pi\colon [n] \to [n] \mid \pi \text{ is a permutation}\}$$

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Degree

$\deg f \leq d$ if f can be written as degree d polynomial in x_{ij}.
$\deg f \leq d$ if f is linear combination of indicators of events

$$\pi(i_1) = j_1, \ldots, \pi(i_d) = j_d.$$

Boolean degree 1 functions on S_n

What are dictators in S_n?

Suppose $f\colon S_n \to \{0,1\}$ has degree 1, i.e.,

$$f = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij}.$$

What does f look like?

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So f is sum of mutually exclusive x_{ij}.

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Two entries are mutually exclusive if on same row or column.
Set of entries is mutually exclusive if all on a single row or column.
Conclusion: f is sum of entries on a single row or column.

Boolean (almost) degree 1 functions on S_n

Dictator theorem (EFP)

If $f\colon S_n \to \{0,1\}$ has degree 1 then
f depends on some $\pi(i)$ or on some $\pi^{-1}(j)$ ("dictator").

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FKN theorem for sparse functions (EFF1)

If $f\colon S_n \to \{0,1\}$ is close to degree 1 and $\mathbb{E}[f] = c/n$ then
f is close to sum of c entries x_{ij}.

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FKN theorem for balanced functions (EFF2)

If $f : S_n \to \{0, 1\}$ is close to degree 1 and $\mathbb{E}[f] \approx 1/2$ then
f is close to a dictator.

What about higher degrees?

Higher-degree analog of dictator theorem

Suppose $f\colon \{0,1\}^n \to \{0,1\}$ has degree d.
On how many coordinates can f depend?

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Higher-degree analog of dictator theorem

Suppose $f\colon \{0,1\}^n \to \{0,1\}$ has degree d.
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Surprising example

Following function has degree d, depends on $\Omega(2^d)$ coordinates:

$$f(x_1,\ldots,x_{d-1},y_0,\ldots,y_{2^{d-1}-1}) = y_x.$$

Can we do better?

Boolean (almost) degree d functions

Nisan–Szegedy theorem, CHS'18

If $f\colon \{0,1\}^n \to \{0,1\}$ has degree d then
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Kindler–Safra theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is close to degree d then f is close to an $O(2^d)$-junta.

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Analogs for slice and S_n

Nisan–Szegedy: known for slice (F.-Ihringer), unknown for S_n.

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Analogs for slice and S_n

Nisan–Szegedy: known for slice (F.-Ihringer), unknown for S_n.
Kindler–Safra: known for slice (FKMW,DFH,KK), known for sparse functions on S_n (EFF3), unknown for balanced functions.

Sparse juntas

Setting: $f\colon \binom{[n]}{k} \to \{0,1\}$, where $p := k/n = o(1)$.

FKN theorem for sparse slice

If f is close to degree 1 then

$$f \text{ or } 1-f \approx g := x_{i_1} + \cdots + x_{i_m}, \quad m = O(1/p).$$

On typical input, ≤ 1 monomials are non-zero, and $g \in \{0,1\}$.

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Sparse junta

g is *sparse junta* if on typical input, $O(1)$ monomials are non-zero, and $g \in \{0,1\}$.
g is *hereditarily sparse junta* if g is sparse junta even given $x_{i_1} = \cdots = x_{i_\ell} = 1$ for $\ell = O(1)$.

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$f \approx$ degree $d \implies f \approx$ degree d hereditarily sparse junta.
Moreover, coefficients of sparse junta belong to some finite set.

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Moreover, coefficients of sparse junta belong to some finite set.

Corollary

If f is ϵ-close to degree d then f is $O(\epsilon^{c_d} + p)$-close to constant.

There's much more!

Other results

Highlights:

- Sharp threshold theorems.
- Small set expansion.
- Invariance principle.

Other domains

Highlights:

- Grassmann scheme.
- High-dimensional expanders.
- Locally testable codes.

Sharp and coarse thresholds

$G(n,p)$ = Erdős–Rényi random graph on n vertices, edge prob p.

Two examples

- $\Pr[G(n, \frac{c}{n})$ contains a triangle$] \longrightarrow 1 - e^{-c^3/6}$.
- $\Pr[G(n, \frac{\log n + c}{n})$ is connected$] \longrightarrow e^{-e^{-c}}$.

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Sharp threshold theorem (Friedgut; Bourgain; Hatami)

Monotone graph properties with coarse threshold are approximately local.

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Swift threshold theorem (Friedgut–Kalai; Bourgain–Kalai)

Monotone graph properties have window size $\tilde{O}(\frac{1}{\log^2 n})$.

Invariance principle

Central limit theorem (Berry–Esséen)

If $x_1, \ldots, x_n$ are i.i.d. samples of $U(\{\pm 1\})$ then

$$\mu + a_1 x_1 + \cdots + a_n x_n \sim N(\mu, a_1^2 + \cdots + a_n^2)$$

provided no a_i is too "prominent".

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$$\mu + a_1 x_1 + \cdots + a_n x_n \sim N(\mu, a_1^2 + \cdots + a_n^2)$$

provided no a_i is too "prominent". Equivalently,

$$\mu + a_1 x_1 + \cdots + a_n x_n \sim \mu + a_1 g_1 + \cdots + a_n g_n,$$

where $g_1, \ldots, g_n$ are i.i.d. samples of $N(0, 1)$.

Invariance principle

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Invariance principle (Mossel–O'Donnell–Oleszkiewicz)

Same holds for degree $O(1)$ polynomials $\sum_S a_S x_S$ provided no variable is too influential: for all i,

$$\sum_{S \ni i} a_S^2 \ll \sum_{S \neq \emptyset} a_S^2.$$

Grassmann scheme

Johnson scheme

$J(n,k)$ is set of subsets of $\{1,\ldots,n\}$ of size k.

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Dictator theorem (F.–Ihringer)

If $f\colon J_2(n,k)\to\{0,1\}$ has degree 1 then

$$f \text{ or } 1-f \in \{0,[x\in V],[y\perp V],[x\in V \vee y\perp V]\} \quad (x \not\perp y)$$

Same object known as: Cameron–Liebler line class, tight set, completely regular strength 0 code of covering radius 1.

There's much more!

Other results

Highlights:

- Sharp threshold theorems.
- Small set expansion.
- Invariance principle.

Other domains

Highlights:

- Grassmann scheme.
- High-dimensional expanders.
- Locally testable codes.