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Approximate Polymorphisms

Joint work with
Noam Lifshitz, Dor Minzer, Elchanan Mossel



Yuval Filmus, 12 March 2021, MIT Reading Group

Judgement Aggregation

Defendant is guilty if they have the **means** and the **motive**

	Means?	**Motive?**	**Guilty?**
Ehud	Yes	No	No
Shamgar	No	Yes	No
Deborah	Yes	Yes	Yes
Majority	**Yes**	**Yes**	**No**

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Deborah	Yes	Yes	Yes
Majority	**Yes**	**Yes**	**No**

Inconsistent!

Polymorphisms

Polymorphisms

Given:

Polymorphisms

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Predicate $P \subseteq \{0,1\}^k$

Polymorphisms

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Function $f\colon \{0,1\}^n \to \{0,1\}$

Polymorphisms

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Function f: $\{0,1\}^n \to \{0,1\}$

$$n \updownarrow \overbrace{\begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline 1 & 0 & 1 \\ \hline 1 & 1 & 0 \\ \hline \cdots & \cdots & \cdots \\ \hline 0 & 0 & 1 \\ \hline \end{array}}^{k}$$

Polymorphisms

Given:

Predicate $P \subseteq \{0,1\}^k$

Function f: $\{0,1\}^n \to \{0,1\}$

k

n

0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

All rows satisfy property P

Polymorphisms

Given:

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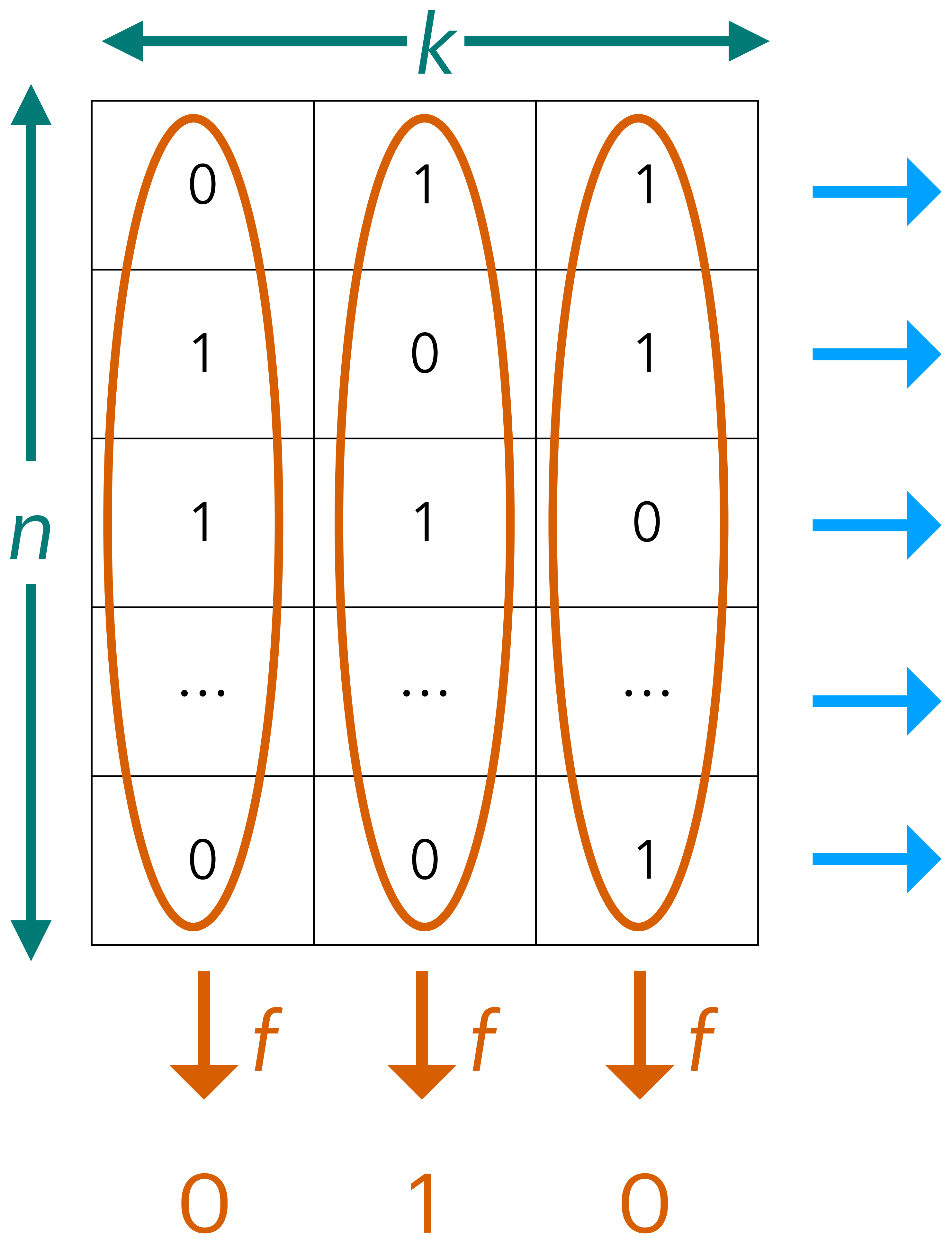

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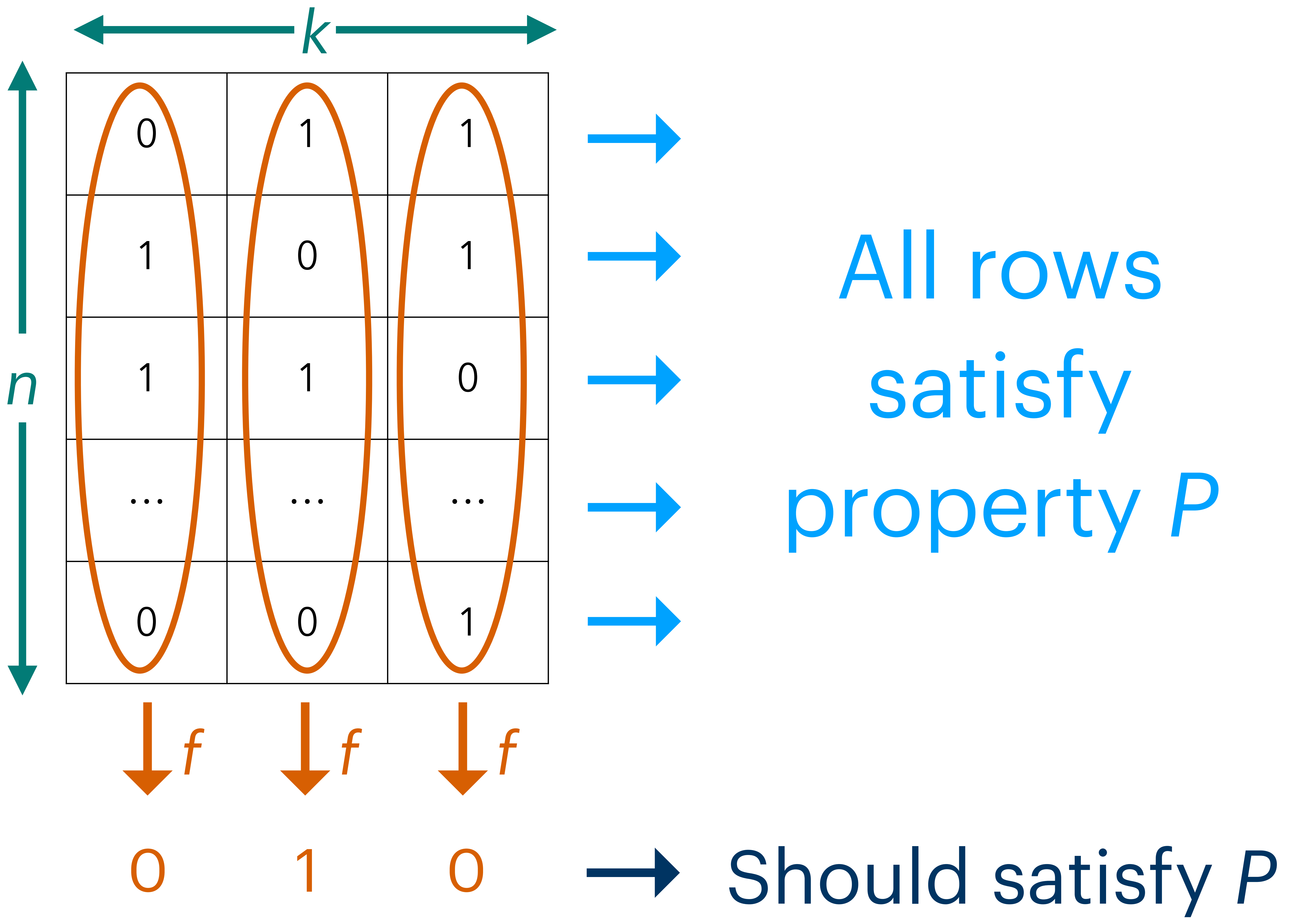


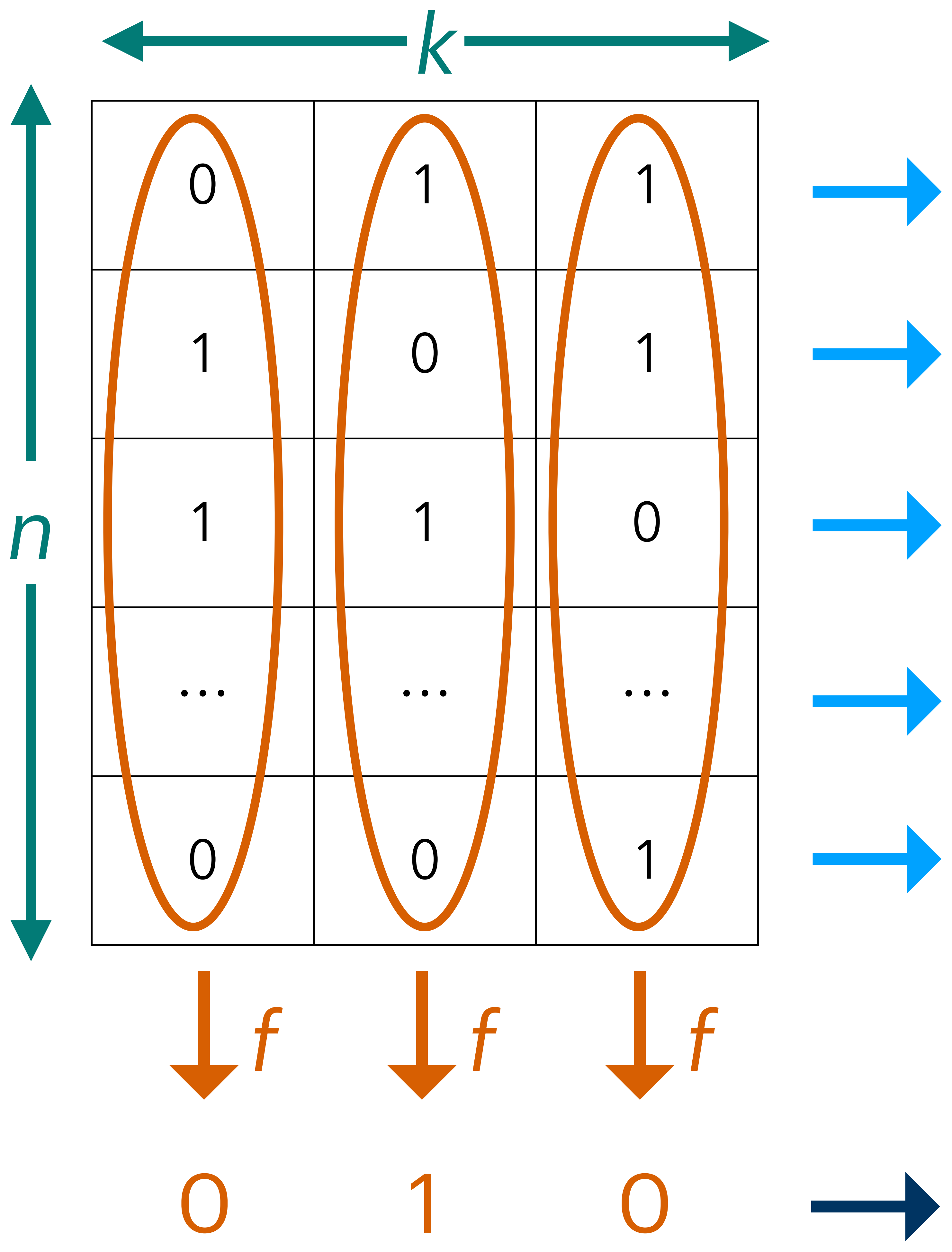
Polymorphisms

Given:

Predicate $P \subseteq \{0,1\}^k$

Function f: $\{0,1\}^n \rightarrow \{0,1\}$

"f is a polymorphism of P"



All rows satisfy property P

Should satisfy P

Example: Not-All-Equal

Voters

A>B?	B>C?	C>A?
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Example: Not-All-Equal

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A>B?	B>C?	C>A?
0	1	1
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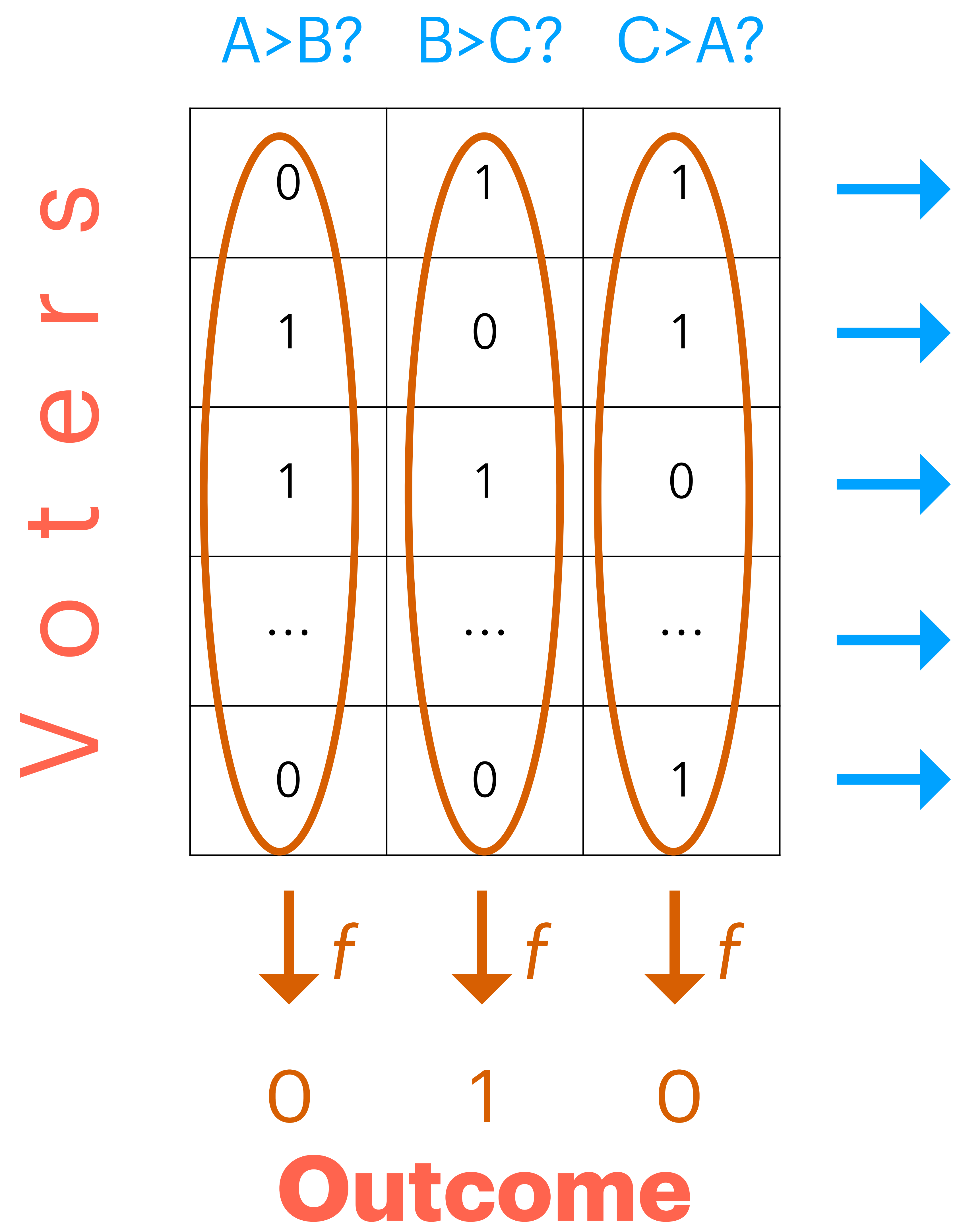
1	1	0
…	…	…
0	0	1

Forbidden rows:

0 0 0

1 1 1

Example: Not-All-Equal



Forbidden rows:
0 0 0
1 1 1

Example: Not-All-Equal

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A>B?	B>C?	C>A?
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

$\downarrow f$ $\downarrow f$ $\downarrow f$

0 1 0

Outcome

Forbidden rows:

0 0 0

1 1 1

Outcome must be legal

Example: Not-All-Equal

Polymorphisms:
Dictators (*i*-th row)
(Arrow's theorem)

Voters

A>B?	B>C?	C>A?
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

$\downarrow f$ $\downarrow f$ $\downarrow f$

0 1 0

Outcome

Forbidden rows:
0 0 0
1 1 1

Outcome must be legal

Example: Even Parity

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *linear* if $f(x \oplus y) = f(x) \oplus f(y)$

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Outcome must have even parity

Example: Even Parity

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *linear* if $f(x \oplus y) = f(x) \oplus f(y)$

Polymorphisms:
XORs of rows

x	y	$x \oplus y$
0	1	1
1	0	1
1	1	0
…	…	…
0	0	0
$\downarrow f$	$\downarrow f$	$\downarrow f$
0	1	1

Rows have even parity

Outcome must have even parity

Example: AND

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *multiplicative* if $f(xy) = f(x)f(y)$

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x	y	$x \wedge y$
0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Example: AND

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x	y	$x \wedge y$
0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

$\rightarrow$

$\rightarrow$

$\rightarrow$

$\rightarrow$

$\rightarrow$

Last coord computes AND

Example: AND

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Example: AND

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x	y	$x \wedge y$	
0	1	0	$\rightarrow$
1	0	0	$\rightarrow$
1	1	1	$\rightarrow$
…	…	…	$\rightarrow$
0	0	0	$\rightarrow$

Last coord computes AND

$\downarrow f$ $\quad$ $\downarrow f$ $\quad$ $\downarrow f$

0 $\quad$ 1 $\quad$ 0 $\rightarrow$ Same condition

Example: AND

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Polymorphisms:
ANDs of rows, 0

x	y	$x \wedge y$
0	1	0
1	0	0
1	1	1
…	…	…
0	0	0
$\downarrow f$	$\downarrow f$	$\downarrow f$
0	1	0

$\rightarrow$ Last coord computes AND

$\rightarrow$ Same condition

Example: NAND

0	1
1	0
0	0
…	…
0	1

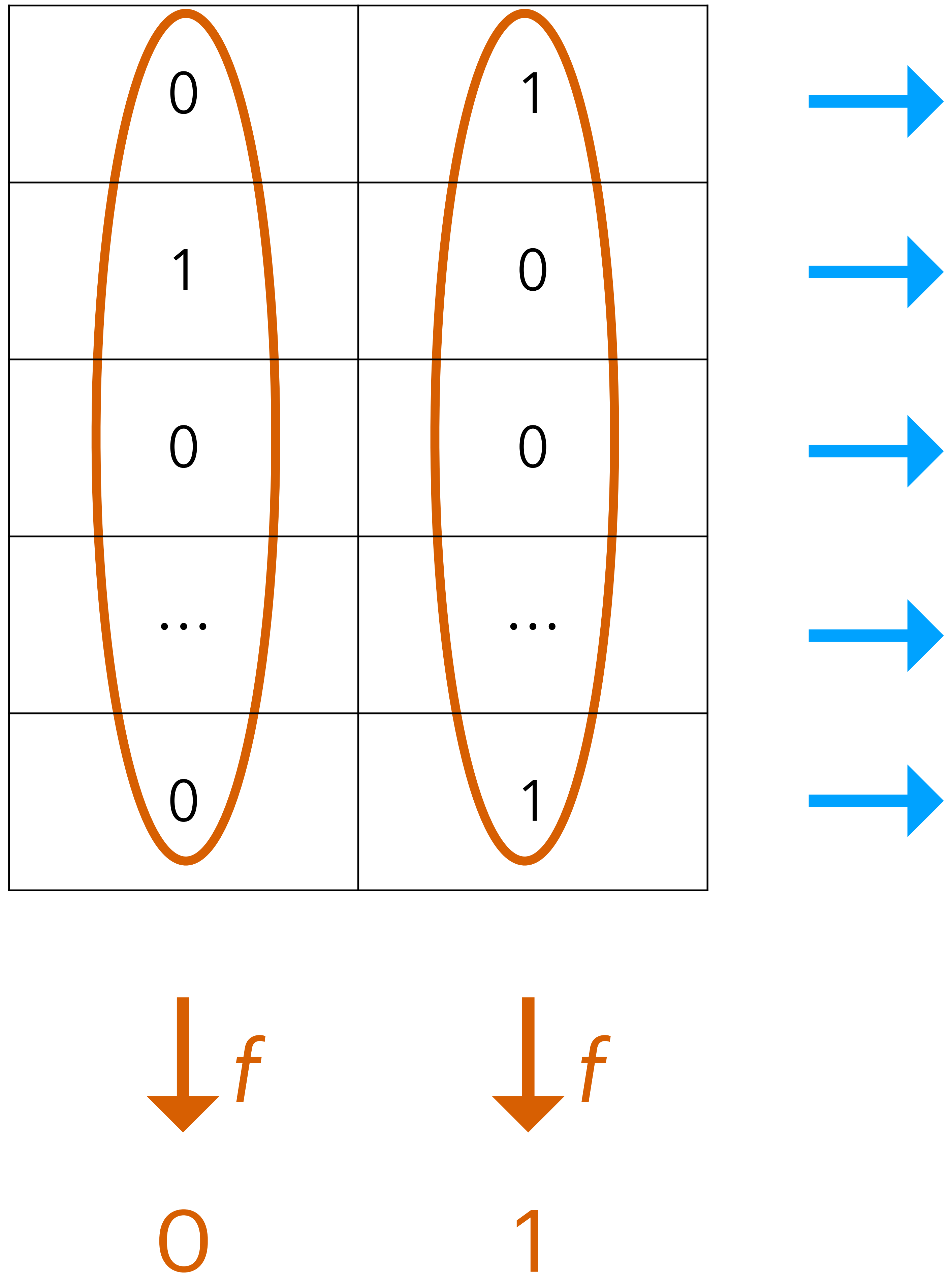
Example: NAND

0	1
1	0
0	0
…	…
0	1

→
→
→
→
→

Forbidden row:
11

Example: NAND



Forbidden row:
11

Example: NAND

0	1
1	0
0	0
$\cdots$	$\cdots$
0	1

$\downarrow f$ $\downarrow f$

0 1

Forbidden row: 11

Outcome not 11

Example: NAND

Polymorphisms:
Intersecting families

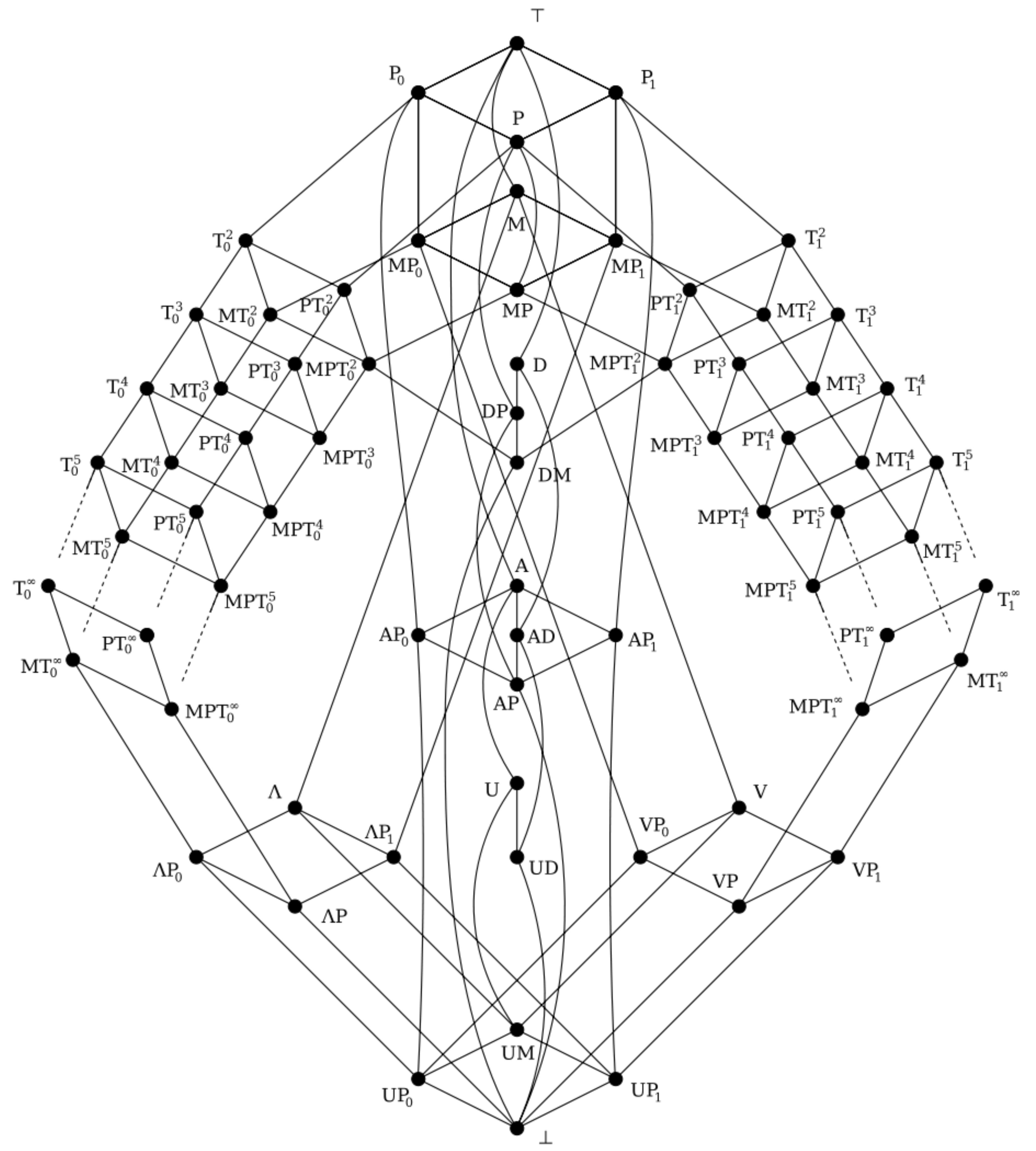
0	1
1	0
0	0
…	…
0	1

$\downarrow f$	$\downarrow f$
0	1

Forbidden row:
11

$\rightarrow$ Outcome not 11

Post’s Lattice



Truth-Functional Setting

Truth-Functional Setting

XOR function

0	1	$0 \oplus 1$
1	0	$1 \oplus 0$
1	1	$1 \oplus 1$
…	…	…
0	0	$0 \oplus 0$

Truth-Functional Setting

XOR function

0	1	$0 \oplus 1$
1	0	$1 \oplus 0$
1	1	$1 \oplus 1$
…	…	…
0	0	$0 \oplus 0$

AND function

0	1	$0 \wedge 1$
1	0	$1 \wedge 0$
1	1	$1 \wedge 1$
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AND function

0	1	$0 \wedge 1$
1	0	$1 \wedge 0$
1	1	$1 \wedge 1$
…	…	…
0	0	$0 \wedge 0$

Majority function

0	1	1	Maj(0,1,1)
1	1	1	Maj(1,1,1)
1	0	0	Maj(1,0,0)
…	…	…	…
0	0	0	Maj(0,0,0)

Truth-Functional Setting

XOR function

0	1	$0 \oplus 1$
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Always have dictators, sometimes "antidictators," sometimes constants

Truth-Functional Setting

XOR function

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0	0	0	Maj(0,0,0)

Always have dictators, sometimes "antidictators," sometimes constants

Dokow & Holzman: Other polymorphisms exist only for AND, XOR

Approximate Polymorphisms

Approximate Polymorphisms

0	1	1
1	0	1
1	1	0
...	...	...
0	0	1

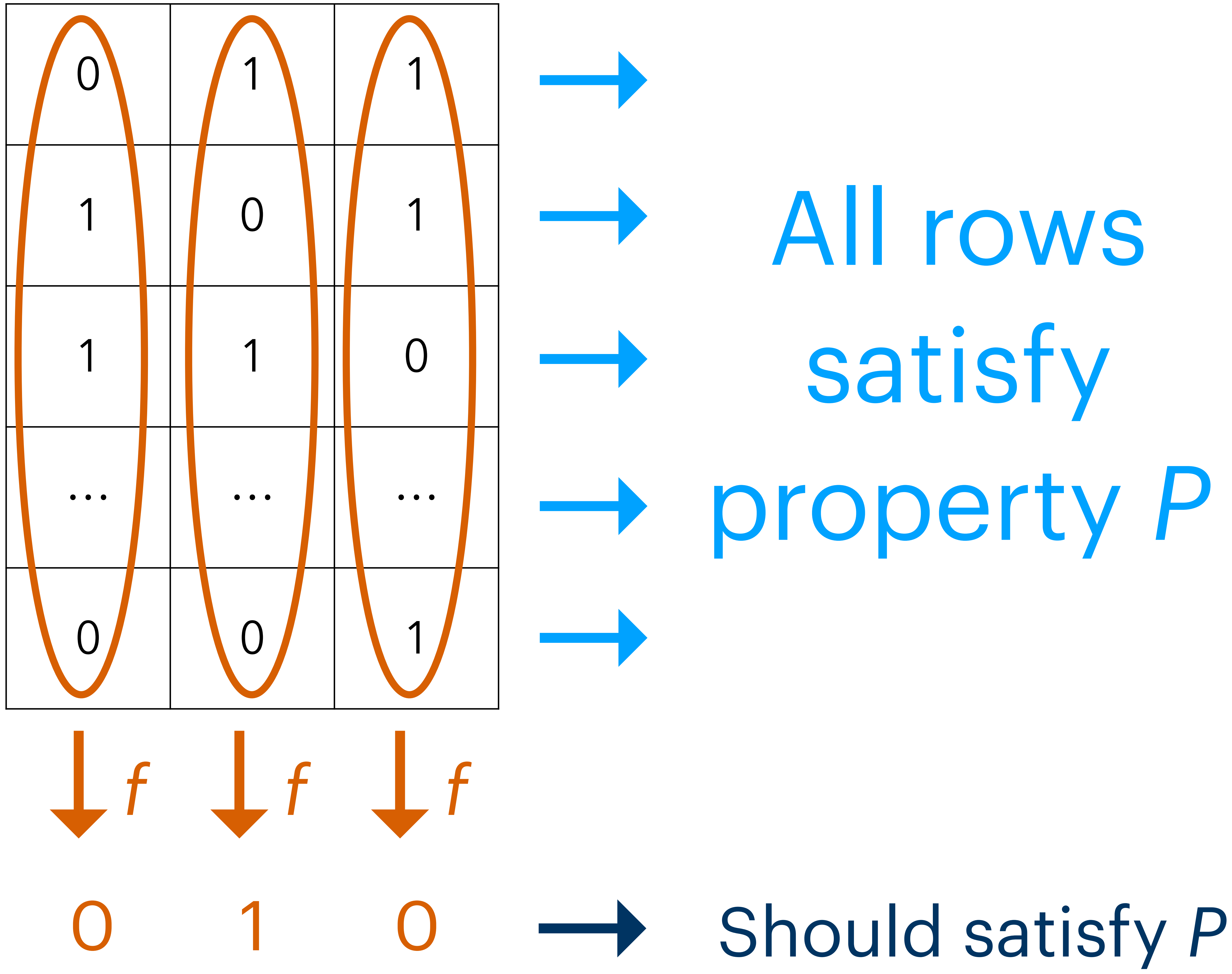
All rows satisfy property P

$\downarrow f$ $\downarrow f$ $\downarrow f$

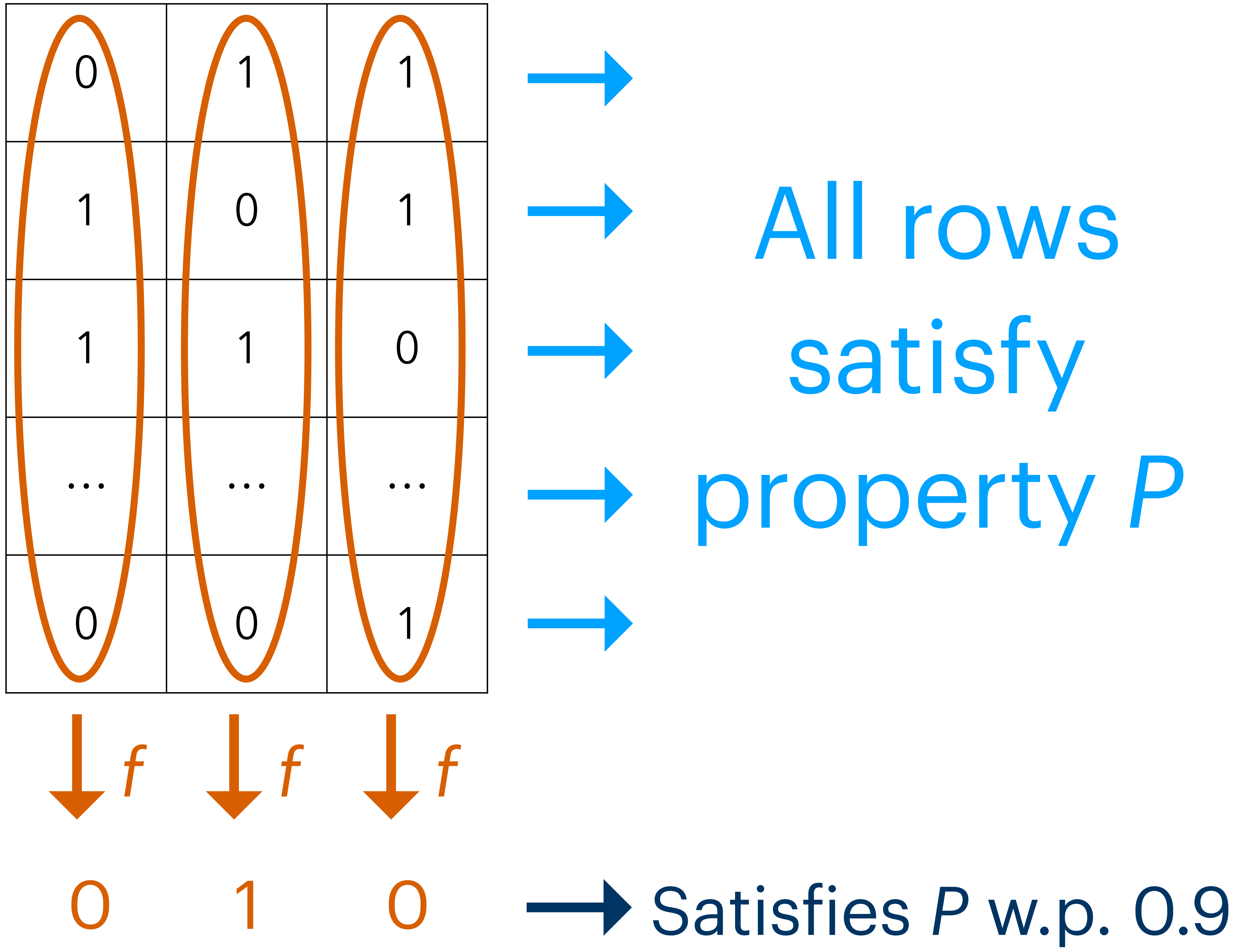
0 1 0 $\rightarrow$ Should satisfy P

Exact polymorphism

Approximate Polymorphisms



Exact polymorphism



Approx polymorphism

Approximate Polymorphisms

0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

$\downarrow f$ $\downarrow f$ $\downarrow f$

0 1 0

All rows satisfy property *P*

Satisfies *P* w.p. 0.9

Is every approximate polymorphism close to an exact polymorphism?

Exact polymorphism

Approx polymorphism

Examples of approximate polymorphisms

Not-All-Equal

0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Even Parity

0	1	1
1	0	1
1	1	0
…	…	…
0	0	0

NAND

0	1
1	0
0	0
…	…
0	1

AND function

0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

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…	…	…
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NAND

0	1
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0	0
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AND function

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Approx polymorphisms:
Dictators (i-th row)

(Kalai's theorem)

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XORs of rows

(Linearity testing)

NAND

0	1
1	0
0	0
…	…
0	1

AND function

0	1	0
1	0	0
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0	1
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Approx polymorphisms:
Intersecting families

(Friedgut–Regev)

AND function

0	1	0
1	0	0
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0	1	1
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Approx polymorphisms:
XORs of rows

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NAND

0	1
1	0
0	0
…	…
0	1

Approx polymorphisms:
Intersecting families

(Friedgut–Regev)

AND function

0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Approx polymorphisms:
ANDs of rows, constant 0

(This work)

Examples of approximate polymorphisms

Not-All-Equal

0	1	1
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1	1	0
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Approx polymorphisms:
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Approx polymorphisms:
XORs of rows

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NAND

0	1
1	0
0	0
…	…
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Approx polymorphisms:
Intersecting families

(Friedgut–Regev)

AND function

0	1	0
1	0	0
1	1	1
…	…	
0	0	0

Improves on Nehama 2010

Approx polymorphisms:
ANDs of rows, constant 0

(This work)

Application to Property Testing

Michal Parnas, Dana Ron, Alex Samorodnitsky: Testing Basic Boolean Formulae (2002)

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Test for f: $\{0,1\}^n \to \{0,1\}$ being a degree k monomial:

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2. Test that $f(x \wedge y) = f(x) \wedge f(y)$.

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[PRS02] could not analyze this “natural” test.

Our work shows that this test works!

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Every function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has unique representation as multilinear poly

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Constant coefficient is expectation of f

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Every function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has unique representation as multilinear poly

Degree of f: degree of unique representation (as polynomial)

Noise operator T_ρ multiplies degree d monomials (“level d“) by ρ^d

Constant coefficient is expectation of f

Important observation: different monomials are *orthogonal*

Simpler example

Majority function

0	1	1	1
1	1	1	1
1	0	0	0
…	…	…	…
0	0	0	0

Last coord computes Majority

Polymorphisms:
Dictators (*i*-th row)
Constant functions

Polymorphisms of Majority

A function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is a polymorphism of Majority if

$$f(\mathsf{Maj}(x_1, y_1, z_1), \ldots, \mathsf{Maj}(x_n, y_n, z_n)) = \mathsf{Maj}(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), f(z_1, \ldots, z_n))$$

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Fix x, average over y, z: $$T_{1/2}f(x) = \frac{1-\mu^2}{2}f(x) + \mu, \qquad \mu = \mathbb{E}[f]$$

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Comparing expectations on both sides: $\mu \in \{0, \pm 1\}$.

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If $\mu = 0$ then $T_{1/2} f = \frac{1}{2} f$, so $\deg f = 1$, so f is a dictator.

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If $\mu \in \{\pm 1\}$, function is constant.

If $\mu = 0$ then $T_{1/2} f = \frac{1}{2} f$, so $\deg f = 1$, so f is a dictator.

Everything also holds approximately, using FKN theorem!

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$$T_{\downarrow}f(x) = \mu f(x), \qquad \mu = \mathbb{E}[f]$$

Polymorphisms of AND

A function $f\colon \{0,1\}^n \to \{0,1\}$ is a polymorphism of AND if

$$f(xy) = f(x)f(y)$$

Fix x, average over y:

$\mathbb{E}[f(xy)]$ = average of f over values "below" x

$$T_\downarrow f(x) = \mu f(x), \qquad \mu = \mathbb{E}[f]$$

Polymorphisms of AND

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Fix x, average over y:

$\mathbb{E}[f(xy)]$ = average of f over values "below" x

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Cannot directly compare biased and unbiased Fourier expansions!

The two expansions depend on different parts of f.

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Easier than proof in paper!

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- Same for Majority on any odd number of inputs.
- Ongoing work: many more functions!

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  Can we generalize this from Maj to any function other than AND, XOR?
- If $\Pr[f(x \oplus y) = f(x) \oplus f(y)] \geq \frac{1}{2} + \varepsilon$ then f *correlates* with exact polymorphism.
  Does a similar statement hold for AND?

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If all predicates have one of the following polymorphisms, in P:

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Otherwise, NP-complete.

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Recently extended to non-binary domains (Dichotomy Theorem).