AND testing and robust judgment aggregation

Yuval Filmus (Technion)
Noam Lifshitz (HUJI)
Dor Minzer (IAS)
Elchanan Mossel (MIT)

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant agreement No. 802020 – HARMONIC)

Introduction

The accused should be convicted if they have both the means and the motive.
Here is what the three judges had to say:

	Means	Motive	Guilty
Holmes	Yes	No	No
Brandeis	No	Yes	No
Cardozo	Yes	Yes	Yes
Majority	Yes	Yes	No

Oops!

Introduction

- This shows that Majority is not *admissible* for AND.
- A judgment aggregation function f: $\{0,1\}^n \longrightarrow \{0,1\}$ is *admissible for AND* if for all $x,y \in \{0,1\}^n$, we have $f(x \wedge y) = f(x) \wedge f(y)$.
- Which functions are admissible?
  - Dictators: $f(x) = x_i$
  - Constants: $f(x) = 0$, $f(x) = 1$
  - Oligarchies (ANDs): $f(x) = x_1 \wedge \cdots \wedge x_m$

Introduction

Theorem: ANDs and constants are only functions admissible for AND.

Are there other solutions which are admissible whp?
(i.e., $\Pr[f(x \wedge y) = f(x) \wedge f(y)] \approx 1$)

Theorem (Nehama): If f is approx admissible, it is approx an AND:

$\Pr[f(x \wedge y) = f(x) \wedge f(y)] \geq 1-\varepsilon \implies f$ is $O(n\varepsilon)$-close to an AND

Want to remove dependence on *n*!

Arrow’s theorem

An election is being held using ranked ballots. The outcome has to be a ranking as well. The final relative ranking of two candidates should depend only on the voters’ relative rankings of these two candidates (IIA).

	Order	A>B?	B>C?	C>A?
Anthony	A>C>B	Yes	No	No
Brutus	B>A>C	No	Yes	No
Caesar	C>B>A	No	No	Yes
Majority	???	No	No	No

Oops!

Linearity testing

The patient should be declared sane if the sandwich has chocolate or pickles, *but not both*.
Here is what three psychiatrists had to say, based on their observations:

	Chocolate	Pickles	Sane
Freud	Yes	No	Yes
Adler	No	Yes	Yes
Lacan	Yes	Yes	No
Majority	Yes	Yes	Yes

Oops!

Universal Algebraic Interpretation

- In universal algebra, a function admissible for AND is called an *AND polymorphism*.
- Similarly, a function admissible for Arrow is an *NAE* polymorphism (NAE = Not All Equal), and a function admissible for linearity testing is an XOR polymorphism.
- Only polymorphisms of NAE are dictators.
- Only polymorphisms of XOR are XORs.
- We are trying to understand *approximate polymorphisms*.

Truth-functionality

- A set of allowed rows is called *truth-functional* if the last column is a function of the previous ones, and this is the only constraint.
    - Both AND and XOR are truth-functional. NAE isn't.
- Dokow and Holzman showed that in the binary truth-functional setting, AND and XOR (on any number of inputs) are the only interesting cases.
    - In all other cases, the only polymorphisms are dictators and, sometimes, constants.

Property Testing Interpretation

- Linearity testing: to test if f: $\{0,1\}^n \longrightarrow \{0,1\}$ is an XOR, sample random x,y and check $f(x \oplus y) = f(x) \oplus f(y)$.
  - If f is XOR, test always succeeds (“completeness”).
  - If test succeeds whp, f is close to XOR (“soundness”).
- Oligarchy testing: to test if f: $\{0,1\}^n \longrightarrow \{0,1\}$ is an AND, sample random x,y and check $f(x \wedge y) = f(x) \wedge f(y)$.
  - Completeness easy to check, want to prove soundness.
  - Goldreich and Ron (TR20-068): $\tilde{O}(1/\varepsilon)$ test.

Linearity testing

How do we prove soundness?

- Method 1: Self-correction (BLR)
  - For most x,y: $f(x) = f(y) \oplus f(x \oplus y)$.
  - “Guess” correct value at x is majority of $f(y) \oplus f(x \oplus y)$.
- Method 2: Fourier analysis (BCHKS)
  - Express success probability of test using Fourier expansion of f.
  - Deduce f can be approximated by single Fourier character.

AND testing

Given f: $\{0,1\}^n \longrightarrow \{0,1\}$ s.t. $f(xy) = f(x)f(y)$ whp, want to deduce that f is close to an AND.

- Method 1: Self-correction
  - Cannot express $f(x)$ in terms of $f(y),f(xy)$.
    "Information is lost."
- Method 2: Fourier analysis
  - Formula for $\Pr[f(xy) = f(x)f(y)]$ isn't nice any more.
    For linearity testing, lucky that XORs=monomials.

Our approach

Suppose $f(xy) = f(x)f(y)$ w.p. ≈ 1.

- Fix x, and take expectation over y:
  - $T_{\downarrow}f(x) \approx \lambda f(x)$, where $\lambda = \mathbb{E}[f]$, where $T_{\downarrow}f(x)$ is average of $f(z)$ on all values $z \leq x$.
- In total, $T_{\downarrow}f \approx \lambda f$ (in appropriate norm).
- Determine approximate eigenvectors of $T_{\downarrow}$.
  - Uses low pass effect of $T_{\downarrow}$ via Bourgain's junta theorem.

Open problems

1. Improve dependence on ε from quasi-poly to poly.
2. Generalize to arbitrary truth-functional setting.
   - In all remaining cases, answer should be dictator.
   - Known for Arrow's theorem using Fourier analysis (Kalai).
3. "List-decoding" version:
   - What if $\Pr[f(x \wedge y) = f(x) \wedge f(y)]$ is better than random?
     - If $\Pr[f(x \oplus y) = f(x) \oplus f(y)] > 1/2$ then f correlates with some XOR.