Intersecting families
and Hoffman’s bound

Yuval Filmus
Technion, Israel

6 December 2021
Outline

1. Introduction: Erdős–Ko–Rado
2. Hoffman’s bound and $t$-intersecting families
3. Uniqueness for intersecting families of permutations
4. Extensions and open problems
Erdős–Ko–Rado theorem

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is *intersecting* if any two sets in $\mathcal{F}$ intersect (are not disjoint).
Erdős–Ko–Rado theorem

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is intersecting if any two sets in $\mathcal{F}$ intersect (are not disjoint).

Theorem

Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting.

- Upper bound:
  If $k \leq n/2$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
Erdős–Ko–Rado theorem

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is *intersecting* if any two sets in $\mathcal{F}$ intersect (are not disjoint).

**Theorem**

**Erdős–Ko–Rado theorem (1938, 1961):**
Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting.

- **Upper bound:**
  If $k \leq n/2$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

- **Uniqueness:**
  If $k < n/2$ and $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F} = \{ S : i \in S \}$ for some $i \in [n]$. 
A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is *intersecting* if any two sets in $\mathcal{F}$ intersect (are not disjoint).

**Theorem**

**Erdős–Ko–Rado theorem (1938, 1961):**
Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting.

- **Upper bound:**
  If $k \leq n/2$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

- **Uniqueness:**
  If $k < n/2$ and $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F} = \{ S : i \in S \}$ for some $i \in [n]$.

- **Stability (Hilton, Milner 1967; Frankl 1987):**
  If $k < n/2$ and $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F} \approx \{ S : i \in S \}$ for some $i \in [n]$. 
Erdős–Ko–Rado theorem

Theorem

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k < n/2$ then

- **Upper bound:** $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- **Uniqueness:** If $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a star (all sets containing $i$).
- **Stability:** If $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F}$ is close to a star.
Erdős–Ko–Rado theorem

**Theorem**

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k < n/2$ then

- **Upper bound**: $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- **Uniqueness**: If $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a star (all sets containing $i$).
- **Stability**: If $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F}$ is close to a star.

Many different proofs:

- Shifting (Erdős, Ko, Rado 1961).
## Erdős–Ko–Rado theorem

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k &lt; n/2$ then</td>
</tr>
<tr>
<td>- <strong>Upper bound:</strong> $</td>
</tr>
<tr>
<td>- <strong>Uniqueness:</strong> If $</td>
</tr>
<tr>
<td>- <strong>Stability:</strong> If $</td>
</tr>
</tbody>
</table>

Many different proofs:

- Shifting (Erdős, Ko, Rado 1961).
- Katona’s circle method (Katona 1972).
Erdős–Ko–Rado theorem

**Theorem**

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k < n/2$ then

- **Upper bound:** $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- **Uniqueness:** If $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a star (all sets containing $i$).
- **Stability:** If $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F}$ is close to a star.

Many different proofs:

- Shifting (Erdős, Ko, Rado 1961).
- Katona’s circle method (Katona 1972).
- Random walk method (Frankl 1978).
### Erdős–Ko–Rado theorem

#### Theorem

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k < n/2$ then

- **Upper bound:** $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- **Uniqueness:** If $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a star (all sets containing $i$).
- **Stability:** If $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F}$ is close to a star.

#### Many different proofs:

- Shifting (Erdős, Ko, Rado 1961).
- Katona's circle method (Katona 1972).
- Random walk method (Frankl 1978).
- Spectral method (Lovász 1979).
Erdős–Ko–Rado theorem

Theorem

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k < n/2$ then

- **Upper bound:** $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- **Uniqueness:** If $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a star (all sets containing $i$).
- **Stability:** If $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F}$ is close to a star.

Many different proofs:

- Shifting (Erdős, Ko, Rado 1961).
- Katona’s circle method (Katona 1972).
- Random walk method (Frankl 1978).
- Spectral method (Lovász 1979).
- Polynomial method (Füredi, Hwang, Weichsel 2006).
Erdős–Ko–Rado theorem

**Theorem**

If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting and $k < n/2$ then

- **Upper bound:** $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- **Uniqueness:** If $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a star (all sets containing $i$).
- **Stability:** If $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F}$ is close to a star.

Many different proofs:

- Shifting (Erdős, Ko, Rado 1961).
- Katona’s circle method (Katona 1972).
- Random walk method (Frankl 1978).
- **Spectral techniques** (Lovász 1979).
- Polynomial method (Füredi, Hwang, Weichsel 2006).
Spectral proof

Kneser graph $K(n, k)$:
- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$. 
Spectral proof

Kneser graph $K(n, k)$:

- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:

- Degree and maximal eigenvalue: $\binom{n-k}{k}$.
- Eigenvalues: $(-1)^j \binom{n-k-j}{k-j}$; minimal: $-(n-k-1)$.
Spectral proof

Kneser graph $K(n, k)$:

- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:

- Degree and maximal eigenvalue: $\binom{n-k}{k}$.
- Eigenvalues: $(-1)^j \binom{n-k-j}{k-j}$; minimal: $-(\binom{n-k-1}{k-1})$.

Proof of upper bound:

- Let $A$ be adjacency matrix of $K(n, k)$. 

Yuval Filmus Technion, Israel
Spectral proof

Kneser graph $K(n, k)$:
- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:
- Degree and maximal eigenvalue: $\binom{n-k}{k}$.
- Eigenvalues: $(-1)^j \binom{n-k-j}{k-j}$; minimal: $-\binom{n-k-1}{k-1}$.

Proof of upper bound:
- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu \binom{n}{k}$. 
Spectral proof

Kneser graph $K(n, k)$:
- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:
- Degree and maximal eigenvalue: $\binom{n-k}{k}$.
- Eigenvalues: $(-1)^j \binom{n-k-j}{k-j}$; minimal: $-(\binom{n-k-1}{k-1})$.

Proof of upper bound:
- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu\binom{n}{k}$.
- Write $f = 1_\mathcal{F}$ as $f = \mu \mathbf{1} + f^\perp$, where $f^\perp \perp \mathbf{1}$. 
Spectral proof

Kneser graph $K(n, k)$:
- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:
- Degree and maximal eigenvalue: $\binom{n-k}{k}$.
- Eigenvalues: $\left(-1\right)^j \binom{n-k-j}{k-j}$; minimal: $-\binom{n-k-1}{k-1}$.

Proof of upper bound:
- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu\binom{n}{k}$.
- Write $f = 1_{\mathcal{F}}$ as $f = \mu 1 + f^\perp$, where $f^\perp \perp 1$.
- $\|f^\perp\|^2 = \|f\|^2 - \|\mu 1\|^2 = \mu - \mu^2$, where $\|g\|^2 = \mathbb{E}[g^2]$.
Spectral proof

Kneser graph $K(n, k)$:
- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:
- Degree and maximal eigenvalue: $\binom{n-k}{k}$.
- Eigenvalues: $(-1)^j \binom{n-k-j}{k-j}$; minimal: $-(\binom{n-k-1}{k-1})$.

Proof of upper bound:
- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu\binom{n}{k}$.
- Write $f = 1_{\mathcal{F}}$ as $f = \mu 1 + f^{\perp}$, where $f^{\perp} \perp 1$.
- $\|f^{\perp}\|^2 = \|f\|^2 - \|\mu 1\|^2 = \mu - \mu^2$, where $\|g\|^2 = \mathbb{E}[g^2]$.
- $0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^{\perp}, Af^{\perp} \rangle \geq \mu^2 \binom{n-k}{k} - (\mu - \mu^2) \binom{n-k-1}{k-1}$. 

Yuval Filmus Technion, Israel Intersecting families and Hoffman’s bound 6 December 2021 5 / 22
Spectral proof

Kneser graph $K(n, k)$:
- Vertices: $\binom{[n]}{k}$.
- Edges: $S \sim T$ if $S \cap T = \emptyset$.

Spectral properties:
- Degree and maximal eigenvalue: $\left(\begin{array}{c}n-k \end{array}\right)$.
- Eigenvalues: $(-1)^j \left(\begin{array}{c}n-k-j \end{array}\right)$; minimal: $-(n-k-1)$.

Proof of upper bound:
- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu\binom{n}{k}$.
- Write $f = 1_{\mathcal{F}}$ as $f = \mu \mathbf{1} + f^\perp$, where $f^\perp \perp \mathbf{1}$.
- $\|f^\perp\|^2 = \|f\|^2 - \|\mu \mathbf{1}\|^2 = \mu - \mu^2$, where $\|g\|^2 = \mathbb{E}[g^2]$.
- $0 = \langle f, Af \rangle = \langle \mu \mathbf{1}, A\mu \mathbf{1} \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 \left(\begin{array}{c}n-k \end{array}\right) - (\mu - \mu^2) \left(\begin{array}{c}n-k-1 \end{array}\right)$.
- Arithmetic: $\mu \leq \frac{k}{n}$, so $|\mathcal{F}| \leq \binom{n-1}{k-1}$. □
Spectral proof

Proof of upper bound:

- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu \binom{n}{k}$.
- Write $f = 1_\mathcal{F}$ as $f = \mu \mathbf{1} + f^\perp$.
- $\|f^\perp\|^2 = \|f\|^2 - \|\mu \mathbf{1}\|^2 = \mu - \mu^2$.
- $0 = \langle f, Af \rangle = \langle \mu \mathbf{1}, A\mu \mathbf{1} \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 \binom{n-k}{k} - (\mu - \mu^2) \binom{n-k-1}{k-1}$.
- Arithmetic: $\mu \leq \frac{k}{n}$, so $|\mathcal{F}| \leq \binom{n-1}{k-1}$. □
Spectral proof

Proof of upper bound:

1. Let \( A \) be adjacency matrix of \( K(n, k) \).
2. Let \( \mathcal{F} \) be intersecting family of size \( \mu \binom{n}{k} \).
3. Write \( f = 1_{\mathcal{F}} \) as \( f = \mu 1 + f^\perp \).
4. \( \|f^\perp\|^2 = \|f\|^2 - \|\mu 1\|^2 = \mu - \mu^2 \).
5. \( 0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 \binom{n-k}{k} - (\mu - \mu^2) \binom{n-k-1}{k-1} \).
6. Arithmetic: \( \mu \leq \frac{k}{n} \), so \( |\mathcal{F}| \leq \binom{n-1}{k-1} \).

Proof of uniqueness (\( \mu = \frac{k}{n} \)):

1. \( f^\perp \) must belong to eigenspace of \(-\binom{n-k-1}{k-1}\).
### Spectral proof

Proof of upper bound:

- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu \binom{n}{k}$.
- Write $f = 1_{\mathcal{F}}$ as $f = \mu \mathbf{1} + f^\perp$.
- $\|f^\perp\|^2 = \|f\|^2 - \|\mu \mathbf{1}\|^2 = \mu - \mu^2$.
- $0 = \langle f, Af \rangle = \langle \mu \mathbf{1}, A \mu \mathbf{1} \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 \binom{n-k}{k} - (\mu - \mu^2) \binom{n-k-1}{k-1}$.
- Arithmetic: $\mu \leq \frac{k}{n}$, so $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f^\perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$. 
Spectral proof

Proof of upper bound:

- Let $A$ be adjacency matrix of $K(n, k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu\binom{n}{k}$.
- Write $f = 1_\mathcal{F}$ as $f = \mu \mathbf{1} + f^\perp$.
- $\|f^\perp\|^2 = \|f\|^2 - \|\mu \mathbf{1}\|^2 = \mu - \mu^2$.
- $0 = \langle f, Af \rangle = \langle \mu \mathbf{1}, A\mu \mathbf{1} \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 \binom{n-k}{k} - (\mu - \mu^2) \binom{n-k-1}{k-1}$.
- Arithmetic: $\mu \leq \frac{k}{n}$, so $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f^\perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.
- Since $f$ is Boolean, $f \in \{0, 1, x_i, 1 - x_i\}$. 
Spectral proof

Proof of upper bound:

- Let $A$ be adjacency matrix of $K(n,k)$.
- Let $\mathcal{F}$ be intersecting family of size $\mu \binom{n}{k}$.
- Write $f = 1_{\mathcal{F}}$ as $f = \mu \mathbf{1} + f^\perp$.
- $\|f^\perp\|^2 = \|f\|^2 - \|\mu \mathbf{1}\|^2 = \mu - \mu^2$.
- $0 = \langle f, Af \rangle = \langle \mu \mathbf{1}, A\mu \mathbf{1} \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 \binom{n-k}{k} - (\mu - \mu^2) \binom{n-k-1}{k-1}$.
- Arithmetic: $\mu \leq \frac{k}{n}$, so $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f^\perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.
- Since $f$ is Boolean, $f \in \{0, 1, x_i, 1-x_i\}$.
- Since $\mu = \frac{k}{n}$, $f = x_i$. 

$\square$
Spectral proof

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f^\perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.
- Since $f$ is Boolean, $f \in \{0, 1, x_i, 1-x_i\}$.
- Since $\mu = \frac{k}{n}$, $f = x_i$. □
Spectral proof

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f^\perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$.
- Since $f$ is Boolean, $f \in \{0, 1, x_i, 1-x_i\}$.
- Since $\mu = \frac{k}{n}$, $f = x_i$. $\square$

Proof of stability ($\mu \approx \frac{k}{n}$):

- $f^\perp$ must be close to eigenspace of $-(\binom{n-k-1}{k-1})$ (in $L_2$).
Spectral proof

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f^\perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.
- Since $f$ is Boolean, $f \in \{0, 1, x_i, 1 - x_i\}$.
- Since $\mu = \frac{k}{n}$, $f = x_i$.

Proof of stability ($\mu \approx \frac{k}{n}$):

- $f^\perp$ must be close to eigenspace of $-(\binom{n-k-1}{k-1})$ (in $L_2$).
- $f$ is close to $c_1 x_1 + \cdots + c_n x_n$. 
Spectral proof

Proof of uniqueness ($\mu = \frac{k}{n}$):

- $f \perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
- $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.
- Since $f$ is Boolean, $f \in \{0, 1, x_i, 1-x_i\}$.
- Since $\mu = \frac{k}{n}$, $f = x_i$. \hfill $\square$

Proof of stability ($\mu \approx \frac{k}{n}$):

- $f \perp$ must be close to eigenspace of $-(\binom{n-k-1}{k-1})$ (in $L_2$).
- $f$ is close to $c_1 x_1 + \cdots + c_n x_n$.
- Since $f$ is Boolean, $f$ is close to $\{0, 1, x_i, 1-x_i\}$ (Friedgut–Kalai–Naor 2002; F. 2016).
Spectral proof

Proof of uniqueness ($\mu = \frac{k}{n}$):

1. $f \perp$ must belong to eigenspace of $-(\binom{n-k-1}{k-1})$.
2. $f$ can be expressed as $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$.
3. Since $f$ is Boolean, $f \in \{0, 1, x_i, 1 - x_i\}$.
4. Since $\mu = \frac{k}{n}$, $f = x_i$. □

Proof of stability ($\mu \approx \frac{k}{n}$):

1. $f \perp$ must be close to eigenspace of $-(\binom{n-k-1}{k-1})$ (in $L_2$).
2. $f$ is close to $c_1 x_1 + \cdots + c_n x_n$.
3. Since $f$ is Boolean, $f$ is close to $\{0, 1, x_i, 1 - x_i\}$ (Friedgut–Kalai–Naor 2002; F. 2016).
4. Since $\mu \approx \frac{k}{n}$, $f \approx x_i$. □
Outline

1. Introduction: Erdős–Ko–Rado
2. Hoffman’s bound and $t$-intersecting families
3. Uniqueness for intersecting families of permutations
4. Extensions and open problems
Hoffman’s classical bound (Hoffman 1970)

Theorem

Suppose $G = (V, E)$ is a $d$-regular graph with minimal eigenvalue $\lambda_{\text{min}}$. 
Hoffman’s classical bound (Hoffman 1970)

Theorem

Suppose $G = (V, E)$ is a $d$-regular graph with minimal eigenvalue $\lambda_{\text{min}}$. Same calculation as before shows that

\[ \alpha(G) \leq \frac{-\lambda_{\text{min}}}{d - \lambda_{\text{min}}} |V|. \]
Hoffman’s classical bound (Hoffman 1970)

**Theorem**

Suppose $G = (V, E)$ is a $d$-regular graph with minimal eigenvalue $\lambda_{\text{min}}$. Same calculation as before shows that

$$\alpha(G) \leq \frac{-\lambda_{\text{min}}}{d - \lambda_{\text{min}}} |V|.$$

**Example**

For the Kneser graph $K(n, k)$,

$$d = \binom{n - k}{k}, \quad \lambda_{\text{min}} = -\binom{n - k - 1}{k - 1}.$$
Hoffman’s classical bound (Hoffman 1970)

**Theorem**

Suppose $G = (V, E)$ is a $d$-regular graph with minimal eigenvalue $\lambda_{\text{min}}$. Same calculation as before shows that

$$\alpha(G) \leq \frac{-\lambda_{\text{min}}}{d - \lambda_{\text{min}}} |V|.$$  

**Example**

For the Kneser graph $K(n, k)$,

$$d = \binom{n-k}{k}, \quad \lambda_{\text{min}} = -\binom{n-k-1}{k-1}.$$  

Therefore

$$\alpha(K(n, k)) \leq \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}} + \binom{n-k}{k-1} \binom{n}{k} = \binom{n-1}{k-1}.$$
Weighted Hoffman’s bound (Lovász 1979)

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is $t$-intersecting if any two sets in $\mathcal{F}$ have at least $t$ common elements.
Weighted Hoffman’s bound (Lovász 1979)

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is $t$-intersecting if any two sets in $\mathcal{F}$ have at least $t$ common elements.

**Theorem**

*Wilson (1984):*

Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is $t$-intersecting.

- **Upper bound:**
  
  If $n \geq (t + 1)(k - t + 1)$ then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. 
**Weighted Hoffman’s bound (Lovász 1979)**

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is *$t$-intersecting* if any two sets in $\mathcal{F}$ have *at least* $t$ common elements.

### Theorem

**Wilson (1984):**

Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is $t$-intersecting.

- **Upper bound:**
  
  If $n \geq (t + 1)(k - t + 1)$ then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.

- **Uniqueness:**
  
  If $n > (t + 1)(k - t + 1)$ and $|\mathcal{F}| = \binom{n-t}{k-t}$ then $\mathcal{F}$ is a $t$-star (all sets containing $i_1, \ldots, i_t$).
Weighted Hoffman’s bound (Lovász 1979)

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is \textit{t-intersecting} if any two sets in $\mathcal{F}$ have \textit{at least} $t$ common elements.

\textbf{Theorem}

\textit{Wilson (1984):}
Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is $t$-intersecting.

- \textbf{Upper bound:}
  \begin{align*}
  \text{If } n \geq (t + 1)(k - t + 1) \text{ then } |\mathcal{F}| &\leq \binom{n-t}{k-t}.
  \end{align*}

- \textbf{Uniqueness:}
  \begin{align*}
  \text{If } n > (t + 1)(k - t + 1) \text{ and } |\mathcal{F}| = \binom{n-t}{k-t} \text{ then } \mathcal{F} \text{ is a } t\text{-star (all sets containing } i_1, \ldots, i_t).}
  \end{align*}

(False for smaller $n$.)
Weighted Hoffman’s bound (Lovász 1979)

A collection $\mathcal{F}$ of $k$-subsets of $[n]$ is $t$-intersecting if any two sets in $\mathcal{F}$ have at least $t$ common elements.

**Theorem**

*Wilson (1984)*: 
Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is $t$-intersecting.

- **Upper bound:** 
  If $n \geq (t + 1)(k - t + 1)$ then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.

- **Uniqueness:** 
  If $n > (t + 1)(k - t + 1)$ and $|\mathcal{F}| = \binom{n-t}{k-t}$ then $\mathcal{F}$ is a $t$-star (all sets containing $i_1, \ldots, i_t$).

*(False for smaller $n$.)

Hoffman’s classical bound gives the wrong bound!
Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, $\alpha(G) \leq \frac{-\lambda_{\min}}{d-\lambda_{\min}} |V|$:

- $A = \text{adjacency matrix}$, $\mathcal{F} = \text{independent set}$, $f = 1_{\mathcal{F}}$. 

Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, \( \alpha(G) \leq \frac{-\lambda_{\text{min}}}{d-\lambda_{\text{min}}} |V| \):

- \( A = \) adjacency matrix, \( \mathcal{F} = \) independent set, \( f = 1_\mathcal{F} \).
- Write \( f = \mu 1 + f^\perp \), where \( \mu = |\mathcal{F}|/|V| \).
Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, \( \alpha(G) \leq \frac{-\lambda_{\text{min}}}{d-\lambda_{\text{min}}} |V| \):

- \( A = \) adjacency matrix, \( \mathcal{F} = \) independent set, \( f = 1_{\mathcal{F}} \).
- Write \( f = \mu 1 + f^\perp \), where \( \mu = |\mathcal{F}|/|V| \).
- \( 0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 d + (\mu - \mu^2) \lambda_{\text{min}} \).
Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, \( \alpha(G) \leq \frac{-\lambda_{\min}}{d-\lambda_{\min}} |V| \):

- \( A = \) adjacency matrix, \( \mathcal{F} = \) independent set, \( f = 1_{\mathcal{F}} \).
- Write \( f = \mu 1 + f^\perp \), where \( \mu = |\mathcal{F}|/|V| \).
- \( 0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 d + (\mu - \mu^2)\lambda_{\min} \).

**Observation**

Suffices for \( A \) to satisfy:

- \( A \) is symmetric.
Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, $\alpha(G) \leq \frac{-\lambda_{\text{min}}}{d - \lambda_{\text{min}}} |V|$:  

- $A =$ adjacency matrix, $\mathcal{F} =$ independent set, $f = 1_{\mathcal{F}}$.
- Write $f = \mu 1 + f^\perp$, where $\mu = |\mathcal{F}|/|V|$.
- $0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 d + (\mu - \mu^2)\lambda_{\text{min}}$.

Observation

Suffices for $A$ to satisfy:

- $A$ is symmetric.
- $A1 = \lambda_1 1$ ($\lambda_1$ replaces $d$).
Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, $\alpha(G) \leq \frac{-\lambda_{\text{min}}}{d-\lambda_{\text{min}}} |V|$: 

- $A = \text{adjacency matrix}$, $\mathcal{F} = \text{independent set}$, $f = 1_{\mathcal{F}}$. 
- Write $f = \mu 1 + f^\perp$, where $\mu = |\mathcal{F}|/|V|$. 
- $0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 d + (\mu - \mu^2)\lambda_{\text{min}}$.

Observation

Suffices for $A$ to satisfy:

- $A$ is symmetric. 
- $A1 = \lambda_1 1$ ($\lambda_1$ replaces $d$). 
- $A(x, y) = 0$ if $(x, y) \notin E$. 
Weighted Hoffman’s bound

Recall proof of Hoffman’s bound, $\alpha(G) \leq \frac{-\lambda_{\min}}{d-\lambda_{\min}} |V|$:

- $A$ = adjacency matrix, $F$ = independent set, $f = 1_F$.
- Write $f = \mu 1 + f^\perp$, where $\mu = |F|/|V|$.
- $0 = \langle f, Af \rangle = \langle \mu 1, A\mu 1 \rangle + \langle f^\perp, Af^\perp \rangle \geq \mu^2 d + (\mu - \mu^2) \lambda_{\min}$.

Observation

Suffices for $A$ to satisfy:

- $A$ is symmetric.
- $A1 = \lambda_1 1$ ($\lambda_1$ replaces $d$).
- $A(x, y) = 0$ if $(x, y) \notin E$.

Resulting bound: $\alpha(G) \leq \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}} |V|$.
Weighted Hoffman’s bound

Theorem (Hoffman’s bound)

Suppose $G = (V, E)$. If a $V \times V$ symmetric matrix satisfies

- $A1 = \lambda_1 1$.
- $A(x, y) = 0$ if $(x, y) \notin E$.

Then $\alpha(G) \leq \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}} |V|$.
Weighted Hoffman’s bound

**Theorem (Hoffman’s bound)**

Suppose $G = (V, E)$. If a $V \times V$ symmetric matrix satisfies

- $A \mathbf{1} = \lambda_1 \mathbf{1}$.
- $A(x, y) = 0$ if $(x, y) \notin E$.

Then $\alpha(G) \leq \frac{-\lambda_{\text{min}}}{\lambda_1 - \lambda_{\text{min}}} |V|$.

**Example (t-intersecting families)**

- By symmetry, $A(S, T)$ depends only on $|S \cap T|$.
Weighted Hoffman’s bound

Theorem (Hoffman’s bound)

Suppose $G = (V, E)$. If a $V \times V$ symmetric matrix satisfies

- $A1 = \lambda_1 1$.
- $A(x, y) = 0$ if $(x, y) \notin E$.

Then $\alpha(G) \leq \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}} |V|$.

Example ($t$-intersecting families)

- By symmetry, $A(S, T)$ depends only on $|S \cap T|$.
- So $A$ belongs to the Bose–Mesner algebra of the Johnson scheme.
Weighted Hoffman’s bound

**Theorem (Hoffman’s bound)**

Suppose $G = (V, E)$. If a $V \times V$ symmetric matrix satisfies

- $A1 = \lambda_1 \cdot 1$.
- $A(x, y) = 0$ if $(x, y) \notin E$.

Then $\alpha(G) \leq \frac{-\lambda_{\text{min}}}{\lambda_1 - \lambda_{\text{min}}} |V|$.

**Example (t-intersecting families)**

- By symmetry, $A(S, T)$ depends only on $|S \cap T|$.
- So $A$ belongs to the Bose–Mesner algebra of the Johnson scheme.
- Analysis must be tight on $t$-stars $\implies V^1, \ldots, V^t$ must be $\lambda_{\text{min}}$-e.s.
Weighted Hoffman’s bound

**Theorem (Hoffman’s bound)**

Suppose \( G = (V, E) \). If a \( V \times V \) symmetric matrix satisfies

- \( A1 = \lambda_1 1 \).
- \( A(x, y) = 0 \) if \( (x, y) \notin E \).

Then \( \alpha(G) \leq \frac{-\lambda_{\text{min}}}{\lambda_1 - \lambda_{\text{min}}} |V| \).

**Example (\( t \)-intersecting families)**

- By symmetry, \( A(S, T) \) depends only on \( |S \cap T| \).
- So \( A \) belongs to the Bose–Mesner algebra of the Johnson scheme.
- Analysis must be tight on \( t \)-stars \( \implies V^1, \ldots, V^t \) must be \( \lambda_{\text{min}} \)-e.s.
- Choose \( \lambda_1 \) arbitrarily \( \implies \) determine \( \lambda_{\text{min}} \) \( \implies \) determine \( A \).
Weighted Hoffman’s bound

**Theorem (Hoffman’s bound)**

Suppose $G = (V, E)$. If a $V \times V$ symmetric matrix satisfies

- $A1 = \lambda_1 1$.
- $A(x, y) = 0$ if $(x, y) \notin E$.

Then $\alpha(G) \leq \frac{-\lambda_{\text{min}}}{\lambda_1 - \lambda_{\text{min}}} |V|$.

**Example (t-intersecting families)**

- By symmetry, $A(S, T)$ depends only on $|S \cap T|$.
- So $A$ belongs to the Bose–Mesner algebra of the Johnson scheme.
- Analysis must be tight on $t$-stars $\implies V^1, \ldots, V^t$ must be $\lambda_{\text{min}}$-e.s.
- Choose $\lambda_1$ arbitrarily $\implies$ determine $\lambda_{\text{min}}$ $\implies$ determine $A$.
- $\lambda_{\text{min}}$ is indeed minimal eigenvalue precisely for correct values of $n$!
Weighted Hoffman’s bound

Theorem (Hoffman’s bound)

Suppose $G = (V, E)$. If a $V \times V$ symmetric matrix satisfies

- $A1 = \lambda_1 1$.
- $A(x, y) = 0$ if $(x, y) \notin E$.

Then $\alpha(G) \leq \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}} |V|$.

Example ($t$-intersecting families)

- By symmetry, $A(S, T)$ depends only on $|S \cap T|$.
- So $A$ belongs to the Bose–Mesner algebra of the Johnson scheme.
- Analysis must be tight on $t$-stars $\implies V^1, \ldots, V^t$ must be $\lambda_{\min}$-e.s.
- Choose $\lambda_1$ arbitrarily $\implies$ determine $\lambda_{\min}$ $\implies$ determine $A$.
- $\lambda_{\min}$ is indeed minimal eigenvalue precisely for correct values of $n$!

Closely related to Lovász $\theta$ function and Delsarte’s LP bound.
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- $t$-intersecting vector spaces (Frankl, Wilson 1986).
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- $t$-intersecting vector spaces (Frankl, Wilson 1986).
- $t$-intersecting permutations (Ellis, Friedgut, Pipel 2011; Ellis 2012; Meagher, Razafimahatratra 2020; Behajaina, Maleki, Rasoamanana, Razafimahatratra 2021+).
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- $t$-intersecting vector spaces (Frankl, Wilson 1986).
- $t$-intersecting permutations (Ellis, Friedgut, Pipel 2011; Ellis 2012; Meagher, Razafimahatratra 2020; Behajaina, Maleki, Rasoamanana, Razafimahatratra 2021+).
- $t$-intersecting perfect matchings (Lindzey 2018; Fallat, Meagher, Shirazi 2021+).
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- $t$-intersecting vector spaces (Frankl, Wilson 1986).
- $t$-intersecting permutations (Ellis, Friedgut, Pipel 2011; Ellis 2012; Meagher, Razafimahatratra 2020; Behajaina, Maleki, Rasoamanana, Razafimahatratra 2021+).
- $t$-intersecting perfect matchings (Lindzey 2018; Fallat, Meagher, Shirazi 2021+).
- Clique-intersecting families of graphs (Ellis, F., Friedgut 2010; Berger, Zhao 2021).
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- $t$-intersecting vector spaces (Frankl, Wilson 1986).
- $t$-intersecting permutations (Ellis, Friedgut, Pipel 2011; Ellis 2012; Meagher, Razafimahatratra 2020; Behajaina, Maleki, Rasoamanana, Razafimahatratra 2021+).
- $t$-intersecting perfect matchings (Lindzey 2018; Fallat, Meagher, Shirazi 2021+).
- Clique-intersecting families of graphs (Ellis, F., Friedgut 2010; Berger, Zhao 2021).
- Many, many more!
  (see Godsil, Meagher 2015 for an exposition of some of these)
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- \( t \)-intersecting vector spaces (Frankl, Wilson 1986).
- \( t \)-intersecting permutations (Ellis, Friedgut, Pipel 2011; Ellis 2012; Meagher, Razafimahatratra 2020; Behajaina, Maleki, Rasoamanana, Razafimahatratra 2021+).
- \( t \)-intersecting perfect matchings (Lindzey 2018; Fallat, Meagher, Shirazi 2021+).
- Clique-intersecting families of graphs (Ellis, F., Friedgut 2010; Berger, Zhao 2021).
- Many, many more!
  (see Godsil, Meagher 2015 for an exposition of some of these)

But: Wilson’s result on \( t \)-intersecting families can be proved by shifting.
More examples

Hoffman’s bound is only known way to prove many intersection theorem:

- $t$-intersecting vector spaces (Frankl, Wilson 1986).
- $t$-intersecting permutations (Ellis, Friedgut, Pipel 2011; Ellis 2012; Meagher, Razafimahatratra 2020; Behajaina, Maleki, Rasoamanana, Razafimahatratra 2021+).
- $t$-intersecting perfect matchings (Lindzey 2018; Fallat, Meagher, Shirazi 2021+).
- Clique-intersecting families of graphs (Ellis, F., Friedgut 2010; Berger, Zhao 2021).
- Many, many more!
  (see Godsil, Meagher 2015 for an exposition of some of these)

But: Wilson’s result on $t$-intersecting families can be proved by shifting. Ahlswede, Khachatrian (1997, 1999): optimal bound for all $n$, $k$, $t$. 
Outline

1. Introduction: Erdős–Ko–Rado
2. Hoffman’s bound and $t$-intersecting families
3. Uniqueness for intersecting families of permutations
4. Extensions and open problems
Intersecting families of permutations

Two permutations \(\alpha, \beta \in S_n\) intersect if \(\alpha(i) = \beta(i)\) for some \(i\).
Intersecting families of permutations

Two permutations $\alpha, \beta \in S_n$ intersect if $\alpha(i) = \beta(i)$ for some $i$.

**Theorem**

If $\mathcal{F} \subseteq S_n$ is intersecting then $|\mathcal{F}| \leq (n - 1)!$.  

Yuval Filmus
Technion, Israel
Intersecting families and Hoffman's bound
6 December 2021
Intersecting families of permutations

Two permutations $\alpha, \beta \in S_n$ intersect if $\alpha(i) = \beta(i)$ for some $i$.

**Theorem**

If $\mathcal{F} \subseteq S_n$ is intersecting then $|\mathcal{F}| \leq (n - 1)!$.

**Proof 1** (Deza, Frankl 1977).

$n$ cyclic shifts of any permutation are pairwise non-intersecting.
Intersecting families of permutations

Two permutations $\alpha, \beta \in S_n$ intersect if $\alpha(i) = \beta(i)$ for some $i$.

**Theorem**

If $\mathcal{F} \subseteq S_n$ is intersecting then $|\mathcal{F}| \leq (n - 1)!$.

**Proof 1 (Deza, Frankl 1977).**

$n$ cyclic shifts of any permutation are pairwise non-intersecting.

**Proof 2 (Renteln 2007).**

Hoffman’s classical bound on the derangement graph.
Intersecting families of permutations

Two permutations $\alpha, \beta \in S_n$ intersect if $\alpha(i) = \beta(i)$ for some $i$.

Theorem

If $\mathcal{F} \subseteq S_n$ is intersecting then $|\mathcal{F}| \leq (n - 1)!$.

Proof 1 (Deza, Frankl 1977).

$n$ cyclic shifts of any permutation are pairwise non-intersecting.

Proof 2 (Renteln 2007).

Hoffman’s classical bound on the derangement graph.

What do extremal families look like?
Intersecting families of permutations

Stars \( \{ \pi \in S_n : \pi(i) = j \} \) are intersecting.
Intersecting families of permutations

Stars \( \{ \pi \in S_n : \pi(i) = j \} \) are intersecting.

**Theorem (Cameron, Ku 2003)**

*Stars are the unique intersecting families of size \((n − 1)!\).*
Intersecting families of permutations

Stars \( \{\pi \in S_n : \pi(i) = j\} \) are intersecting.

Theorem (Cameron, Ku 2003)

Stars are the unique intersecting families of size \((n - 1)!\).

Proof (Ellis, Friedgut, Pilpel 2011).
- Hoffman’s bound implies that extremal families have degree 1.

Definition (Terminology)
- **Degree**: minimal degree of poly in \(x_{ij} = 1_{\pi(i)=j}\) agreeing with \(f\).
Intersecting families of permutations

Stars \( \{ \pi \in S_n : \pi(i) = j \} \) are intersecting.

**Theorem (Cameron, Ku 2003)**

*Stars are the unique intersecting families of size \( (n - 1)! \).*

**Proof (Ellis, Friedgut, Pilpel 2011).**

- Hoffman’s bound implies that extremal families have degree 1.
- Degree 1 families are **dictators** (Birkhoff–von Neumann).

**Definition (Terminology)**

- **Degree:** minimal degree of poly in \( x_{ij} = 1_{\pi(i)=j} \) agreeing with \( f \).
- **Dictator:** function depending only on \( \pi(i) \) or only on \( \pi^{-1}(j) \).
Intersecting families of permutations

Stars \( \{ \pi \in S_n : \pi(i) = j \} \) are intersecting.

**Theorem (Cameron, Ku 2003)**

*Stars are the unique intersecting families of size \((n - 1)!\).*

**Proof (Ellis, Friedgut, Pilpel 2011).**

- Hoffman’s bound implies that extremal families have **degree 1**.
- Degree 1 families are **dictators** (Birkhoff–von Neumann).
- Intersecting dictators are stars.

**Definition (Terminology)**

- **Degree**: minimal degree of poly in \( x_{ij} = 1_{\pi(i)=j} \) agreeing with \( f \).
- **Dictator**: function depending only on \( \pi(i) \) or only on \( \pi^{-1}(j) \).
Two permutations $\alpha, \beta \in S_n$ \textit{t-intersect} if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$. 

$t$-intersecting families of permutations
Two permutations $\alpha, \beta \in S_n$ \textit{t-intersect} if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$.

\textbf{Theorem (Ellis, Friedgut, Pilpel 2011)}

If $\mathcal{F} \subseteq S_n$ is \textit{t-intersecting} and $n \geq N_t$ then $|\mathcal{F}| \leq (n - t)!$. 
\[ t \]-intersecting families of permutations

Two permutations \( \alpha, \beta \in S_n \) \( t \)-intersect if \( \alpha(i_j) = \beta(i_j) \) for some \( i_1, \ldots, i_t \).

**Theorem (Ellis, Friedgut, Pilpel 2011)**

If \( \mathcal{F} \subseteq S_n \) is \( t \)-intersecting and \( n \geq N_t \) then \( |\mathcal{F}| \leq (n - t)! \).

Proved using Hoffman's bound and representation theory of \( S_n \).
$t$-intersecting families of permutations

Two permutations $\alpha, \beta \in S_n$ $t$-intersect if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$.

**Theorem (Ellis, Friedgut, Pilpel 2011)**

If $\mathcal{F} \subseteq S_n$ is $t$-intersecting and $n \geq N_t$ then $|\mathcal{F}| \leq (n - t)!$.

Proved using Hoffman’s bound and representation theory of $S_n$.

What about uniqueness?
Two permutations $\alpha, \beta \in S_n$ $t$-intersect if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$.

Theorem (Ellis, Friedgut, Pilpel 2011)

If $\mathcal{F} \subseteq S_n$ is $t$-intersecting and $n \geq N_t$ then $|\mathcal{F}| \leq (n - t)!$.

Proved using Hoffman’s bound and representation theory of $S_n$.

What about uniqueness?

t-stars $\{\pi \in S_n : \pi(i_k) = j_k \text{ for } k \in [t]\}$ are $t$-intersecting.
**t-intersecting families of permutations**

Two permutations $\alpha, \beta \in S_n$ *t-intersect* if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$.

**Theorem (Ellis, Friedgut, Pilpel 2011)**

If $\mathcal{F} \subseteq S_n$ is $t$-intersecting and $n \geq N_t$ then $|\mathcal{F}| \leq (n - t)!$.

Proved using Hoffman’s bound and representation theory of $S_n$.

What about uniqueness?

$t$-stars $\{\pi \in S_n : \pi(i_k) = j_k$ for $k \in [t]\}$ are $t$-intersecting.

**Theorem (Ellis 2011)**

If $\mathcal{F} \subseteq S_n$ is $t$-intersecting, $|\mathcal{F}| = (n - t)!$ and $n \geq N_t$ then $\mathcal{F}$ is a $t$-star.
$t$-intersecting families of permutations

Two permutations $\alpha, \beta \in S_n$ \textit{$t$-intersect} if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$.

**Theorem (Ellis, Friedgut, Pilpel 2011)**

If $F \subseteq S_n$ is $t$-intersecting and $n \geq N_t$ then $|F| \leq (n - t)!$.

Proved using Hoffman’s bound and representation theory of $S_n$.

What about uniqueness?

$t$-stars $\{\pi \in S_n : \pi(i_k) = j_k \text{ for } k \in [t]\}$ are $t$-intersecting.

**Theorem (Ellis 2011)**

If $F \subseteq S_n$ is $t$-intersecting, $|F| = (n - t)!$ and $n \geq N_t$ then $F$ is a $t$-star.

Complicated proof (but proves a much stronger result).
**t-intersecting families of permutations**

Two permutations $\alpha, \beta \in S_n$ *t-intersect* if $\alpha(i_j) = \beta(i_j)$ for some $i_1, \ldots, i_t$.

**Theorem (Ellis, Friedgut, Pilpel 2011)**

If $\mathcal{F} \subseteq S_n$ is $t$-intersecting and $n \geq N_t$ then $|\mathcal{F}| \leq (n - t)!$.

Proved using Hoffman’s bound and representation theory of $S_n$.

**What about uniqueness?**

$t$-stars $\{\pi \in S_n : \pi(i_k) = j_k \text{ for } k \in [t]\}$ are $t$-intersecting.

**Theorem (Ellis 2011)**

If $\mathcal{F} \subseteq S_n$ is $t$-intersecting, $|\mathcal{F}| = (n - t)!$ and $n \geq N_t$ then $\mathcal{F}$ is a $t$-star.

Complicated proof (but proves a much stronger result).

Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_\mathcal{F}$ has degree $t$. 
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_{\mathcal{F}}$ has degree $t$.
Uniqueness for \( t \)-intersecting families of permutations

- Starting point: \( t \)-intersecting family \( \mathcal{F} \) of size \((n - t)!\), where \( n \) large.
- Hoffman’s bound implies that \( 1_{\mathcal{F}} \) has degree \( t \).
- Nisan 1991 + Nisan, Szegedy 1994: \( \mathcal{F} \) has certificate complexity \( K_t \).
  - If \( \pi \in \mathcal{F} \) then \( \exists i_1, \ldots, i_{K_t} \) s.t. \( \rho \in \mathcal{F} \) whenever \( \rho(i_j) = \pi(i_j) \) \( \forall j \in [K_t] \).
Uniqueness for \( t \)-intersecting families of permutations

- Starting point: \( t \)-intersecting family \( \mathcal{F} \) of size \((n - t)!\), where \( n \) large.
- Hoffman’s bound implies that \( 1_\mathcal{F} \) has degree \( t \).
- Nisan 1991 + Nisan, Szegedy 1994: \( \mathcal{F} \) has certificate complexity \( K_t \).
  - If \( \pi \in \mathcal{F} \) then \( \exists i_1, \ldots, i_{K_t} \) s.t. \( \rho \in \mathcal{F} \) whenever \( \rho(i_j) = \pi(i_j) \) \( \forall j \in [K_t] \).
- Suppose \( \mathcal{F} \) is not contained in any \( t \)-star.
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_\mathcal{F}$ has degree $t$.
  - If $\pi \in \mathcal{F}$ then $\exists i_1, \ldots, i_{K_t}$ s.t. $\rho \in \mathcal{F}$ whenever $\rho(i_j) = \pi(i_j) \forall j \in [K_t]$.
- Suppose $\mathcal{F}$ is not contained in any $t$-star.
- Choose a certificate $C_0$ for some arbitrary $\pi_0 \in \mathcal{F}$. 
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_\mathcal{F}$ has degree $t$.
  - If $\pi \in \mathcal{F}$ then $\exists i_1, \ldots, i_{K_t}$ s.t. $\rho \in \mathcal{F}$ whenever $\rho(i_j) = \pi(i_j) \forall j \in [K_t]$.
- Suppose $\mathcal{F}$ is not contained in any $t$-star.
- Choose a certificate $C_0$ for some arbitrary $\pi_0 \in \mathcal{F}$.
- Any $\pi \in \mathcal{F}$ must $t$-intersect $C_0$, say $\pi(i_j) = \pi_0(i_j)$ for $i_1, \ldots, i_t \in C_0$. 

Yuval Filmus  
Technion, Israel  
Intersecting families and Hoffman’s bound  
6 December 2021
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_{\mathcal{F}}$ has degree $t$.
  - If $\pi \in \mathcal{F}$ then $\exists i_1, \ldots, i_{K_t}$ s.t. $\rho \in \mathcal{F}$ whenever $\rho(i_j) = \pi(i_j) \forall j \in [K_t]$.
- Suppose $\mathcal{F}$ is not contained in any $t$-star.
- Choose a certificate $C_0$ for some arbitrary $\pi_0 \in \mathcal{F}$.
- Any $\pi \in \mathcal{F}$ must $t$-intersect $C_0$, say $\pi(i_j) = \pi_0(i_j)$ for $i_1, \ldots, i_t \in C_0$.
- Let $\rho \in \mathcal{F}$ satisfies $\rho(i_j) \neq \pi_0(i_j)$ for some $j \in [t]$. 
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n-t)!$, where $n$ large.
- Hoffman’s bound implies that $1_{\mathcal{F}}$ has degree $t$.
  - If $\pi \in \mathcal{F}$ then $\exists i_1, \ldots, i_{K_t}$ s.t. $\rho \in \mathcal{F}$ whenever $\rho(i_j) = \pi(i_j) \; \forall j \in [K_t]$.
- Suppose $\mathcal{F}$ is not contained in any $t$-star.
- Choose a certificate $C_0$ for some arbitrary $\pi_0 \in \mathcal{F}$.
- Any $\pi \in \mathcal{F}$ must $t$-intersect $C_0$, say $\pi(i_j) = \pi_0(i_j)$ for $i_1, \ldots, i_t \in C_0$.
- Let $\rho \in \mathcal{F}$ satisfies $\rho(i_j) \neq \pi_0(i_j)$ for some $j \in [t]$.
- $\pi$ must $t$-intersect a certificate $C_\rho$ for $\rho$, so $\pi(i_{t+1}) = \rho(i_{t+1})$ for some $i_{t+1} \in C_\rho$ different from $i_1, \ldots, i_t$. 
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_\mathcal{F}$ has degree $t$.
  - If $\pi \in \mathcal{F}$ then $\exists i_1, \ldots, i_{K_t}$ s.t. $\rho \in \mathcal{F}$ whenever $\rho(i_j) = \pi(i_j)$ $\forall j \in [K_t]$.
- Suppose $\mathcal{F}$ is not contained in any $t$-star.
- Choose a certificate $C_0$ for some arbitrary $\pi_0 \in \mathcal{F}$.
- Any $\pi \in \mathcal{F}$ must $t$-intersect $C_0$, say $\pi(i_j) = \pi_0(i_j)$ for $i_1, \ldots, i_t \in C_0$.
- Let $\rho \in \mathcal{F}$ satisfies $\rho(i_j) \neq \pi_0(i_j)$ for some $j \in [t]$.
- $\pi$ must $t$-intersect a certificate $C_\rho$ for $\rho$, so $\pi(i_{t+1}) = \rho(i_{t+1})$ for some $i_{t+1} \in C_\rho$ different from $i_1, \ldots, i_t$.
- Altogether, $\pi$ belongs to some $(t + 1)$-star.
Uniqueness for $t$-intersecting families of permutations

- Starting point: $t$-intersecting family $\mathcal{F}$ of size $(n - t)!$, where $n$ large.
- Hoffman’s bound implies that $1_{\mathcal{F}}$ has degree $t$.
  - If $\pi \in \mathcal{F}$ then $\exists i_1, \ldots, i_{K_t}$ s.t. $\rho \in \mathcal{F}$ whenever $\rho(i_j) = \pi(i_j)$ $\forall j \in [K_t]$.
- Suppose $\mathcal{F}$ is not contained in any $t$-star.
- Choose a certificate $C_0$ for some arbitrary $\pi_0 \in \mathcal{F}$.
- Any $\pi \in \mathcal{F}$ must $t$-intersect $C_0$, say $\pi(i_j) = \pi_0(i_j)$ for $i_1, \ldots, i_t \in C_0$.
- Let $\rho \in \mathcal{F}$ satisfies $\rho(i_j) \neq \pi_0(i_j)$ for some $j \in [t]$.
- $\pi$ must $t$-intersect a certificate $C_\rho$ for $\rho$, so $\pi(i_{t+1}) = \rho(i_{t+1})$ for some $i_{t+1} \in C_\rho$ different from $i_1, \ldots, i_t$.
- Altogether, $\pi$ belongs to some $(t + 1)$-star.
- $\mathcal{F}$ covered by $\binom{K_t}{t} K_t$ many $(t + 1)$-stars $\implies |\mathcal{F}| = O((n - t - 1)!)$. 
Outline

1. Introduction: Erdős–Ko–Rado
2. Hoffman’s bound and $t$-intersecting families
3. Uniqueness for intersecting families of permutations
4. Extensions and open problems
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:

- $s$-wise intersecting families (Frankl–Tokushige 2003, FGL2021+):
  Every $s$ sets intersect.
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:

- $s$-wise intersecting families (Frankl–Tokushige 2003, FGL2021+): Every $s$ sets intersect.
- $s$-wise $t$-intersecting families (unsolved in general!): Every $s$ sets have $t$ elements in common.
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:

- $s$-wise intersecting families (Frankl–Tokushige 2003, FGL2021+): Every $s$ sets intersect.
- $s$-wise $t$-intersecting families (unsolved in general!): Every $s$ sets have $t$ elements in common.
- Erdős matching conjecture (unsolved in general!): No $s$-matching.
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:

- $s$-wise intersecting families (Frankl–Tokushige 2003, FGL2021+):
  Every $s$ sets intersect.
- $s$-wise $t$-intersecting families (unsolved in general!):
  Every $s$ sets have $t$ elements in common.
- Erdős matching conjecture (unsolved in general!):
  No $s$-matching.
- Frankl’s triangle problem (Frankl 1990, FGL2021+):
  No $A, B, C$ s.t. $A \triangle B \triangle C = \emptyset$. 
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:

- $s$-wise intersecting families (Frankl–Tokushige 2003, FGL2021+): Every $s$ sets intersect.
- $s$-wise $t$-intersecting families (unsolved in general!): Every $s$ sets have $t$ elements in common.
- Erdős matching conjecture (unsolved in general!): No $s$-matching.
- Frankl’s triangle problem (Frankl 1990, FGL2021+): No $A, B, C$ s.t. $A \triangle B \triangle C = \emptyset$.
- Mantel’s theorem (FGL2021+): Graphs without triangles.
Versions for hypergraphs

Hoffman’s bound extended to hypergraphs by several authors:

- Golubev 2016.
- Bachoc, Gundert, Passuello 2019.

Independent sets in $s$-uniform hypergraphs correspond to intersection conditions on $s$-tuples:

- $s$-wise intersecting families (Frankl–Tokushige 2003, FGL2021+): Every $s$ sets intersect.
- $s$-wise $t$-intersecting families (unsolved in general!): Every $s$ sets have $t$ elements in common.
- Erdős matching conjecture (unsolved in general!): No $s$-matching.
- Frankl’s triangle problem (Frankl 1990, FGL2021+): No $A, B, C$ s.t. $A \triangle B \triangle C = \emptyset$.
- Mantel’s theorem (FGL2021+): Graphs without triangles.
- Turán problems in hypergraphs (unsolved in general).
Sum of squares hierarchy

- Hoffman’s bound relies on \( A + \lambda_{\text{min}} I \succeq 0 \iff f^T (A + \lambda_{\text{min}} I) f \geq 0 \ \forall f. \)
Sum of squares hierarchy

- Hoffman’s bound relies on $A + \lambda_{\text{min}}I \succeq 0 \iff f^T (A + \lambda_{\text{min}}I)f \geq 0 \ \forall f$.

- Theory of quadratic forms:

  $$f^T(A + \lambda_{\text{min}})f = \sum_i \ell_i^2$$

  for linear functions $\ell_i$ in entries of $f$. 
Sum of squares hierarchy

- Hoffman’s bound relies on $A + \lambda_{\text{min}} I \succeq 0 \iff f^T(A + \lambda_{\text{min}} I)f \geq 0 \ \forall f$.
- Theory of quadratic forms:
  \[ f^T(A + \lambda_{\text{min}})f = \sum_i \ell_i^2 \]
  for linear functions $\ell_i$ in entries of $f$.
- Sum of squares hierarchy: allow $\ell_i$ of higher degree.
Sum of squares hierarchy

- Hoffman’s bound relies on $A + \lambda_{\text{min}}I \succeq 0 \iff f^T (A + \lambda_{\text{min}}I) f \geq 0 \ \forall f$. 
- Theory of quadratic forms:
  
  $$f^T (A + \lambda_{\text{min}}) f = \sum_i \ell_i^2$$

  for linear functions $\ell_i$ in entries of $f$.
- Sum of squares hierarchy: allow $\ell_i$ of higher degree.
- Known to be tight for degree $|V|$!
Sum of squares hierarchy

- Hoffman’s bound relies on $A + \lambda_{\text{min}} I \succeq 0 \iff f^T (A + \lambda_{\text{min}} I) f \geq 0 \ \forall f$.
- Theory of quadratic forms:
  
  $$f^T (A + \lambda_{\text{min}}) f = \sum_i \ell_i^2$$

  for linear functions $\ell_i$ in entries of $f$.
- Sum of squares hierarchy: allow $\ell_i$ of higher degree.
- Known to be tight for degree $|V|$ !
- Gives better bounds on codes for concrete parameters.
Sum of squares hierarchy

- Hoffman’s bound relies on $A + \lambda_{\min} I \succeq 0 \iff f^T (A + \lambda_{\min} I) f \geq 0 \ \forall f$.
- Theory of quadratic forms:
  
  $$f^T (A + \lambda_{\min}) f = \sum_{i} \ell_i^2$$

  for linear functions $\ell_i$ in entries of $f$.
- Sum of squares hierarchy: allow $\ell_i$ of higher degree.
- Known to be tight for degree $|V|$ !
- Gives better bounds on codes for concrete parameters.
- Challenge: Apply to EKR theory.
Sum of squares hierarchy

- Hoffman’s bound relies on $A + \lambda_{\min}I \succeq 0 \iff f^T(A + \lambda_{\min}I)f \geq 0 \forall f$.
- Theory of quadratic forms:
  $$f^T(A + \lambda_{\min})f = \sum_i \ell_i^2$$
  for linear functions $\ell_i$ in entries of $f$.
- Sum of squares hierarchy: allow $\ell_i$ of higher degree.
- Known to be tight for degree $|V|$ !
- Gives better bounds on codes for concrete parameters.
- Challenge: Apply to EKR theory.
  - $t$-intersecting families for $n < (t + 1)(k - t + 1)$?
  (spectral proof of Ahlswede–Khachatrian theorem)
Challenges

- $s$-wise $t$-intersecting families.
Challenges

- s-wise $t$-intersecting families.
- Erdős matching conjecture.
Challenges

- s-wise $t$-intersecting families.
- Erdős matching conjecture.
- Intersecting families of triangulations (Kalai).
Challenges

- s-wise \( t \)-intersecting families.
- Erdős matching conjecture.
- Intersecting families of triangulations (Kalai).
- \( t \)-intersecting families of permutations for all \( n, t \).

Yuval Filmus
Technion, Israel

Intersecting families and Hoffman’s bound
6 December 2021
Challenges

- s-wise $t$-intersecting families.
- Erdős matching conjecture.
- Intersecting families of triangulations (Kalai).
- $t$-intersecting families of permutations for all $n, t$.
- Chvátal's conjecture:
  If $\mathcal{F}$ is downwards-closed family of sets, then maximum size of an intersecting family is attained by some star (not necessarily uniquely).