Intersecting families and Hoffman's bound

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Outline

1. Introduction: Erdős–Ko–Rado
2. Hoffman's bound and t-intersecting families
3. Uniqueness for intersecting families of permutations
4. Extensions and open problems

Erdős–Ko–Rado theorem

A collection $\mathcal{F}$ of k-subsets of $[n]$ is *intersecting* if any two sets in $\mathcal{F}$ intersect (are not disjoint).

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- *Stability (Hilton, Milner 1967; Frankl 1987):*
  If $k < n/2$ and $|\mathcal{F}| \approx \binom{n-1}{k-1}$ then $\mathcal{F} \approx \{S : i \in S\}$ for some $i \in [n]$.

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If $\mathcal{F} \subseteq \binom{[n]}{k}$ *is intersecting and* $k < n/2$ *then*

- *Upper bound:* $|\mathcal{F}| \leq \binom{n-1}{k-1}$.
- *Uniqueness: If* $|\mathcal{F}| = \binom{n-1}{k-1}$ *then* $\mathcal{F}$ *is a star (all sets containing* i*).*
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$$\alpha(K(n,k)) \le \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} \binom{n}{k} = \binom{n-1}{k-1}.$$

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Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ *is* t*-intersecting.*

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Hoffman's classical bound gives the wrong bound!

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Closely related to Lovász θ function and Delsarte's LP bound.

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But: Wilson's result on t-intersecting families can be proved by shifting.
Ahlswede, Khachatrian (1997, 1999): optimal bound for *all* n, k, t.

Outline

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What do extremal families look like?

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Below: simple argument of Dafni, F., Lifshitz, Lindzey, Vinyals 2021.

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- π must t-intersect a certificate C_ρ for ρ, so $\pi(i_{t+1}) = \rho(i_{t+1})$ for some $i_{t+1} \in C_\rho$ different from $i_1, \ldots, i_t$.
- Altogether, π belongs to some $(t+1)$-star.
- $\mathcal{F}$ covered by $\binom{K_t}{t} K_t$ many $(t+1)$-stars $\Longrightarrow |\mathcal{F}| = O((n-t-1)!)$.

Outline

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- Turán problems in hypergraphs (unsolved in general).

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    (spectral proof of Ahlswede–Khachatrian theorem)

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- Chvátal's conjecture:
  If $\mathcal{F}$ is downwards-closed family of sets, then maximum size of an intersecting family is attained by some star (not necessarily uniquely).