Fast matrix multiplication: Limitations of C.–W. method

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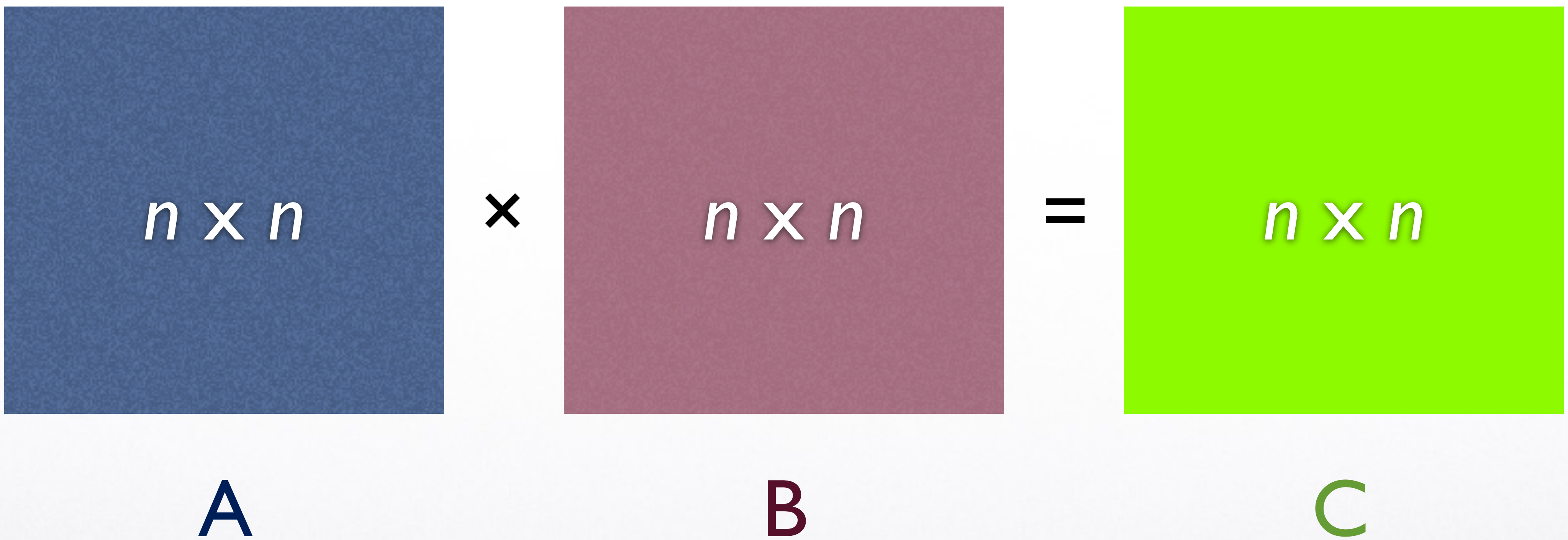
Joint work with Andris Ambainis (U. of Latvia) and François Le Gall (U. of Tokyo)

Matrix multiplication



How fast can we do it?

Why do we care?

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Equivalent to:

- Matrix inverse
- Solving linear equations
- Determinant
- Diagonalization

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Used to solve:

- Testing graphs for triangles
- All pairs shortest paths
- Parsing of context-free languages

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Forms inner loop in:

- Linear programming

High-school algorithm

```
for (i=0; i<n; i++)
  for (j=0; j<n; j++)
    for (k=0; k<n; k++)
      C[i][k] += A[i][j] * B[j][k];
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Complexity: $O(n^3)$

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- Coppersmith & Winograd (1987): $O(n^{2.376})$

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- Strassen (1986): $O(n^{2.48})$
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- Stothers (2010), Williams (2012), Le Gall (2014): $O(n^{2.373})$

Exponent of matrix multiplication

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- Conjecture: $\omega = 2$
- Best lower bound: $\Omega(n^2 \log n)$ [Raz]

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- We show: this method cannot prove $\omega < 2.3725$ (cf. best known bound $\omega < 2.3728$)
- We suggest generalized method which could break this limit (but cannot prove $\omega < 2.3078$)

Proof outline

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- Develop a framework encompassing analysis of all powers at once
- Prove a limitation on new framework

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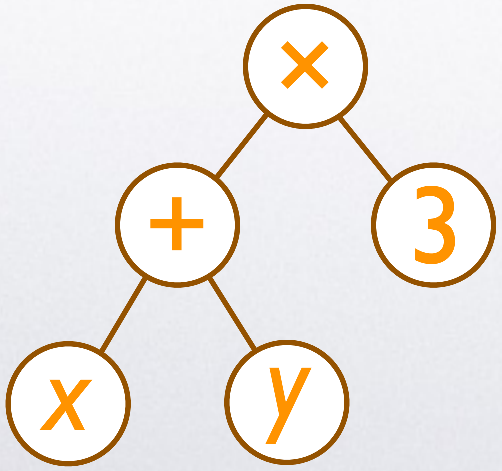
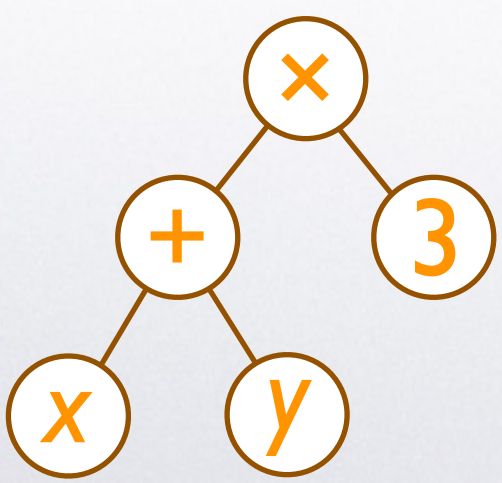


What is computation?

- Different computation models have different power
- Natural model for us – algebraic:
  - Algebraic circuits over +,−,×,÷ with constants
  - Straight-line programs



$t = x + y$

$out = t \times 3$

Bilinear algorithms

Example: $(a + b\,i) \times (c + d\,i) = x + y\,i$

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$$\ell_1 = a + b$$
$$\ell_2 = c + d$$

$$m_1 = a \times c$$
$$m_2 = b \times d$$
$$m_3 = \ell_1 \times \ell_2$$

$$x = m_1 - m_2$$
$$t = m_1 + m_2$$
$$y = m_3 - t$$

Bilinear algorithms

Example: $(a + b\,i) \times (c + d\,i) = x + y\,i$

$$\left.\begin{aligned} \ell_1 &= a + b \\ \ell_2 &= c + d \end{aligned}\right\}$$ Linear combinations of input

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Bilinear algorithms

Example: $(a + b\,i) \times (c + d\,i) = x + y\,i$

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Strassen: normal form for bilinear functions

Strassen's algorithm

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

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$m_5 = (a_{11}+a_{12})(b_{22})$

$m_6 = (a_{21}-a_{11})(b_{11}+b_{12})$

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$c_{11} = m_1+m_4-m_5+m_7$
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$c_{21} = m_2+m_4$
$c_{22} = m_1-m_2+m_3+m_6$

Strassen's algorithm

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

Strassen's algorithm

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$M_1 = (A_{11}+A_{22})(B_{11}+B_{22})$

•

•

•

2x2 algorithm

$M_7 = (A_{12}-A_{22})(B_{21}+B_{22})$

Strassen's algorithm

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•

•

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2x2 algorithm

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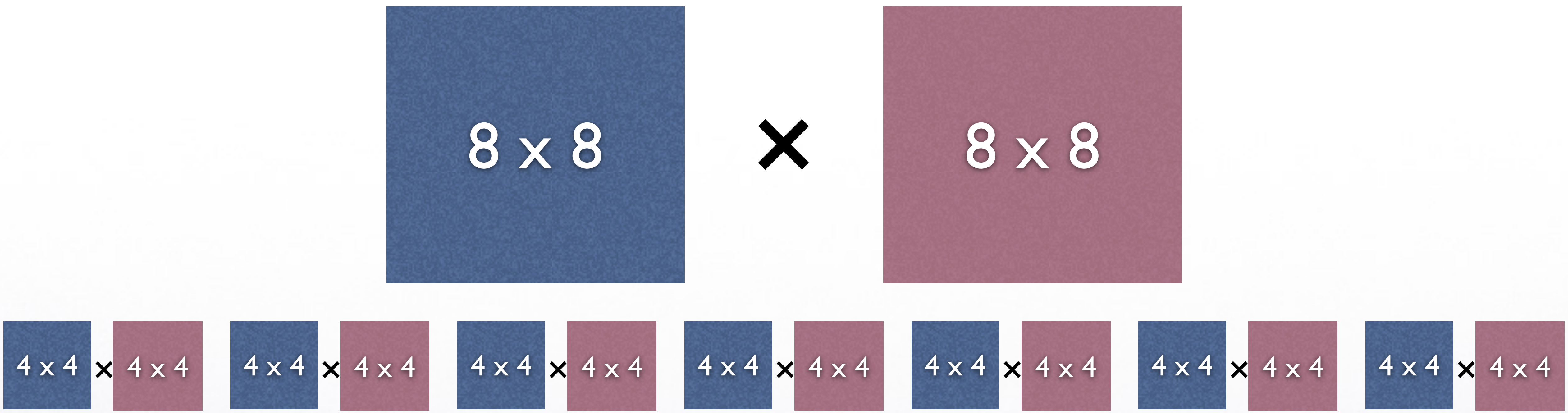
Strassen's algorithm

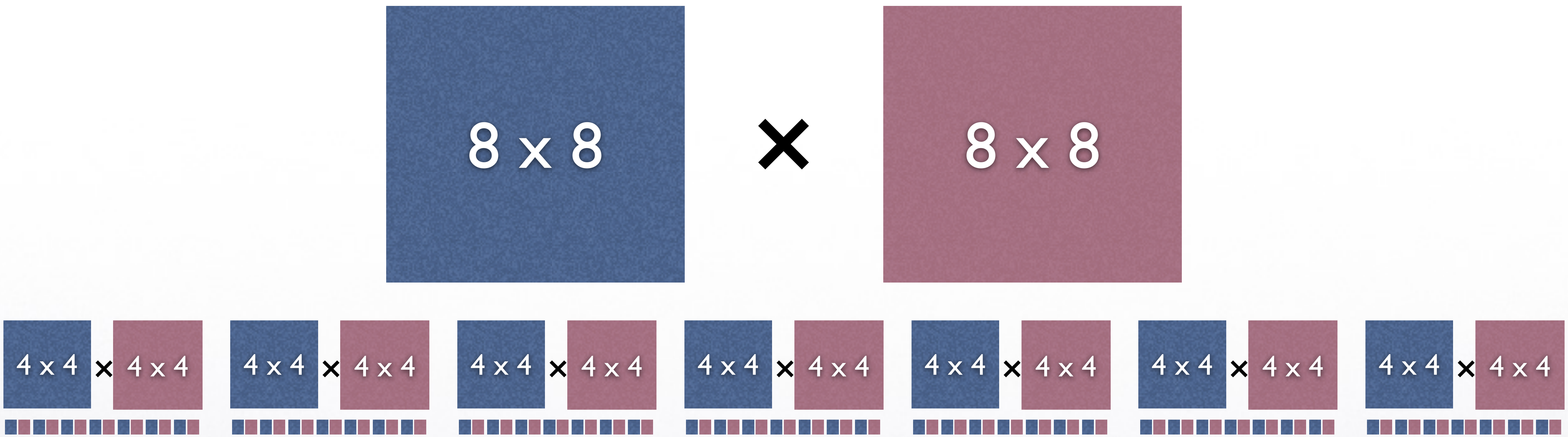
8 x 8 × 8 x 8

Strassen's algorithm



7 multiplications 4x4

Strassen's algorithm



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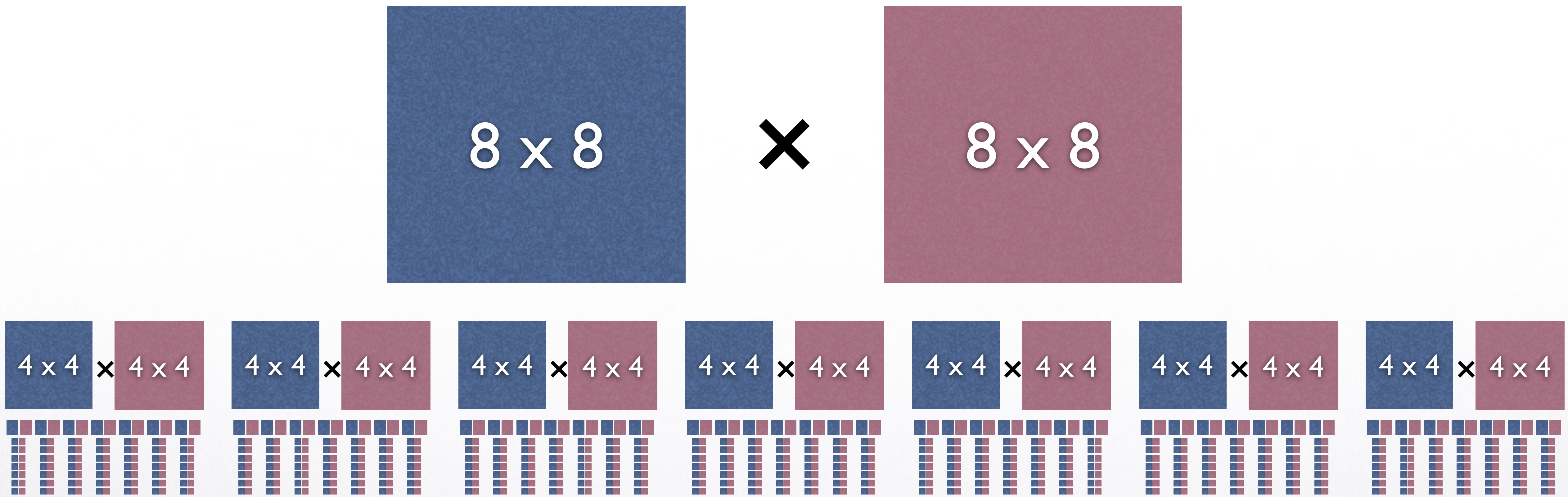

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7x7x7 multiplications 1x1

Strassen’s algorithm

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- Multiplying 2x2 matrices: 7 multiplications

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  $7^d=(2^d)^{\log_2 7}\approx(2^d)^{2.81}$ multiplications

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- Multiplying $n\times n$ matrices: $O(n^{2.81})$

Tensor notation

Strassen's 2x2 algorithm

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$$a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{12}+a_{12}b_{22}c_{12}+$$
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$$=$$
$$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+$$
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Strassen's identity

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$$a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{12}+a_{12}b_{22}c_{12}+ a_{21}b_{11}c_{21}+a_{22}b_{21}c_{21}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22}$$

$$=$$

$$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+ (a_{21}+a_{22})(b_{11})(c_{21}-c_{22})+ (a_{11})(b_{12}-b_{22})(c_{12}+c_{22})+ (a_{22})(b_{21}-b_{11})(c_{11}+c_{21})+ (a_{11}+a_{12})(b_{22})(c_{12}-c_{11})+ (a_{21}-a_{11})(b_{11}+b_{12})(c_{22})+ (a_{12}-a_{22})(b_{21}+b_{22})(c_{11})$$

Strassen's identity

$R(\langle 2,2,2\rangle) \leq 7$

Bilinear view of matrices

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Matrices are bilinear operators:

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Rank one matrices:

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Bilinear view of matrices

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Rank = min no. rank one matrices summing to M

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Row rank = Rank = Column rank

More on tensors

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Tensors: three-dimensional matrices

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$$\langle n,m,p\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} a_{ij}b_{jk}c_{ik}$$

$nm \times mp \times np$ tensor

More on tensors

Tensors: three-dimensional matrices

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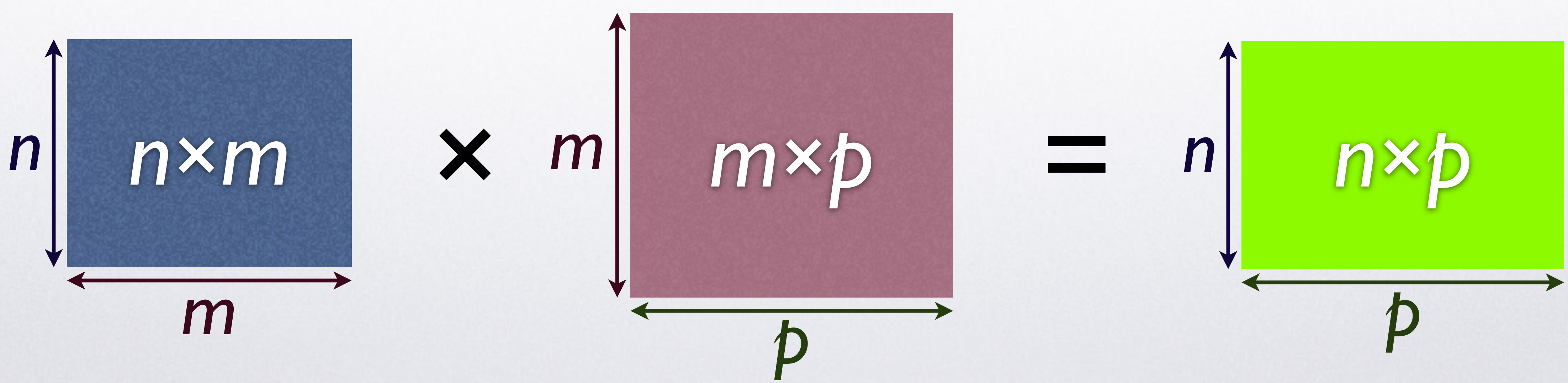
Matrix multiplication tensor



More on tensors

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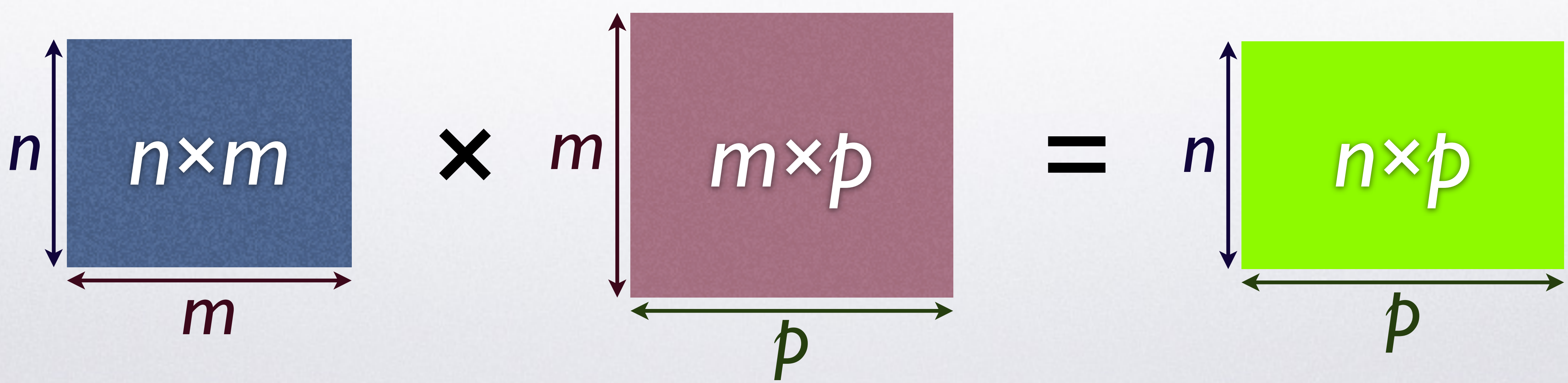
$nm \times mp \times np$ tensor

1 at row (i,j)
column (j,k)
depth (i,k)

Matrix multiplication tensor



Tensor rank

$$a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{12}+a_{12}b_{22}c_{12}+$$
$$a_{21}b_{11}c_{21}+a_{22}b_{21}c_{21}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22}=$$
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Strassen's identity

Tensor rank

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$\Big\}\ \langle 2,2,2\rangle$

$$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+$$
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Strassen's identity

Tensor rank

$$\left.\begin{array}{l} a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{12}+a_{12}b_{22}c_{12}+ \\ a_{21}b_{11}c_{21}+a_{22}b_{21}c_{21}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22}= \end{array}\right\} \langle 2,2,2\rangle$$

$$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+$$
$$(a_{21}+a_{22})(b_{11})(c_{21}-c_{22})+$$
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Rank one tensors

Strassen's identity

Tensor rank

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$a_{21}b_{11}c_{21}+a_{22}b_{21}c_{21}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22}=$ $\Big\}\ \langle 2,2,2\rangle$

$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+$
$(a_{21}+a_{22})(b_{11})(c_{21}-c_{22})+$
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Strassen's identity

$R(T)$ = tensor rank of T: min no. rank one tensors summing to T

Tensor rank

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$\Big\}$ $\langle 2,2,2\rangle$

$$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+$$
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Strassen's identity

R(T) = tensor rank of T:
min no. rank one tensors summing to T

NP-hard to compute! (Håstad)

Tensor rank

$$a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{12}+a_{12}b_{22}c_{12}+$$
$$a_{21}b_{11}c_{21}+a_{22}b_{21}c_{21}+a_{21}b_{12}c_{22}+a_{22}b_{22}c_{22}=$$

$\Big\}$ $\langle 2,2,2\rangle$

$$(a_{11}+a_{22})(b_{11}+b_{22})(c_{11}+c_{22})+$$
$$(a_{21}+a_{22})(b_{11})(c_{21}-c_{22})+$$
$$(a_{11})(b_{12}-b_{22})(c_{12}+c_{22})+$$
$$(a_{22})(b_{21}-b_{11})(c_{11}+c_{21})+$$
$$(a_{11}+a_{12})(b_{22})(c_{12}-c_{11})+$$
$$(a_{21}-a_{11})(b_{11}+b_{12})(c_{22})+$$
$$(a_{12}-a_{22})(b_{21}+b_{22})(c_{11})$$

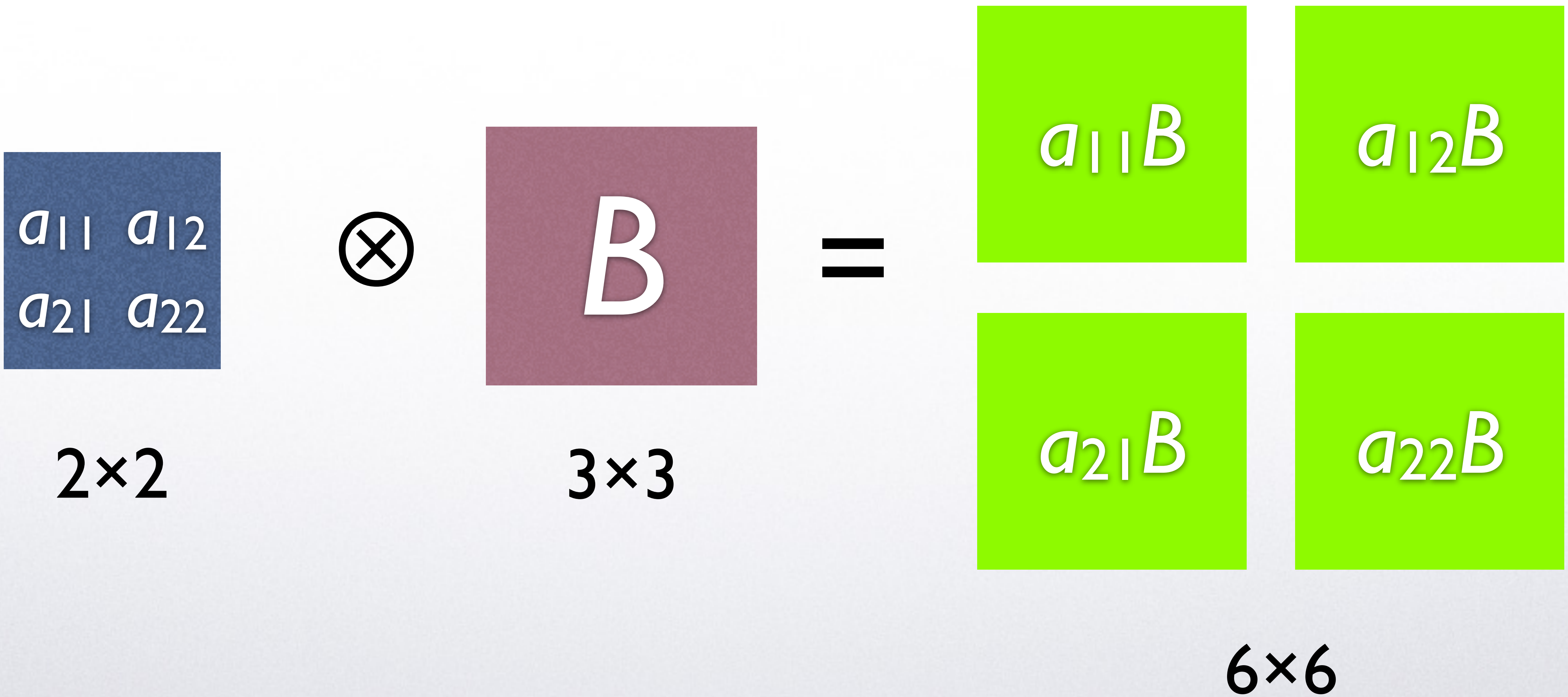
R(T) = tensor rank of T: min no. rank one tensors summing to T

NP-hard to compute! (Håstad)

Strassen's identity ⟶ $R(\langle 2,2,2\rangle)\leq 7$

Tensor product

Kronecker product of matrices:



Tensor product

Kronecker product of matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

2×2 $\quad$ 3×3 $\quad$ 6×6

Tensor product of tensors – 3D analog

Tensor product

Tensor product

- Example: $\langle n_1,m_1,p_1\rangle\otimes\langle n_2,m_2,p_2\rangle=\langle n_1n_2,m_1m_2,p_1p_2\rangle$

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$$R(\langle 2^d,2^d,2^d\rangle)=R(\langle 2,2,2\rangle^{\otimes d})\leq R(\langle 2,2,2\rangle)^d=7^d$$

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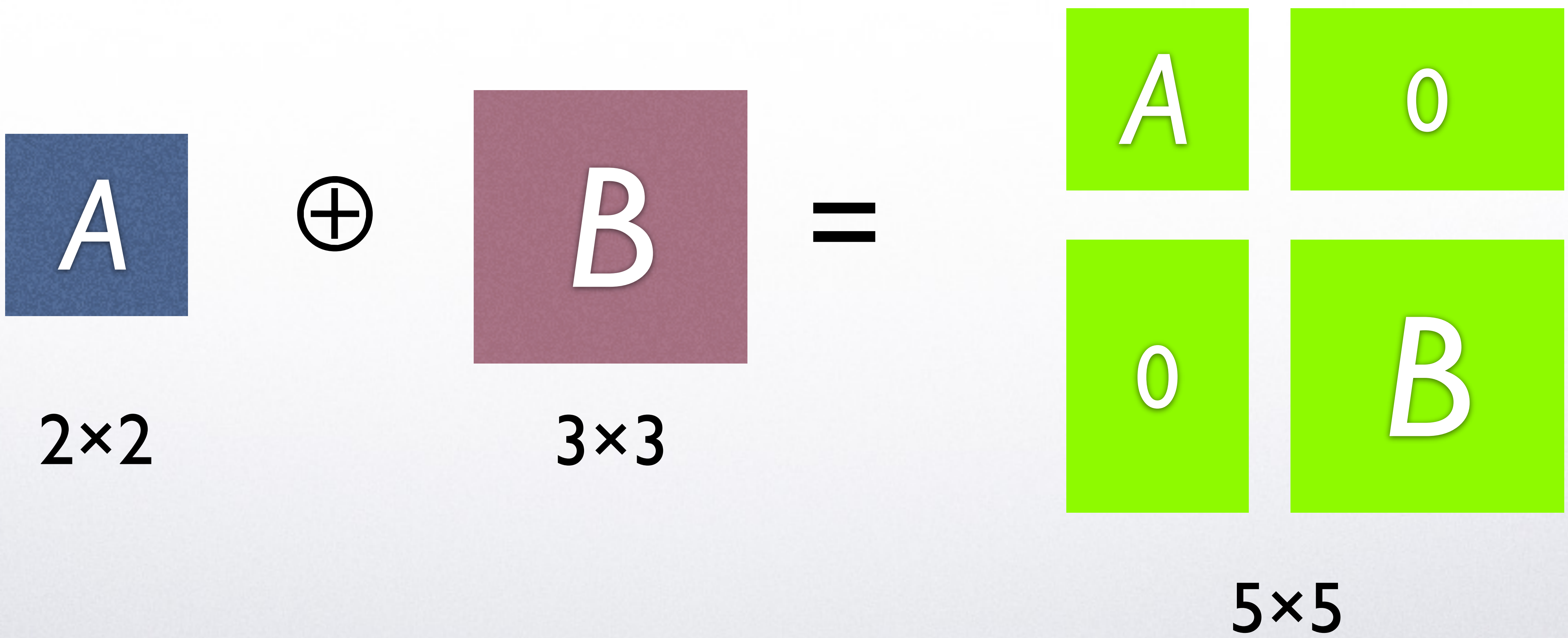
- Algebraic definition of ω

Tensor sum

Direct sum of matrices:



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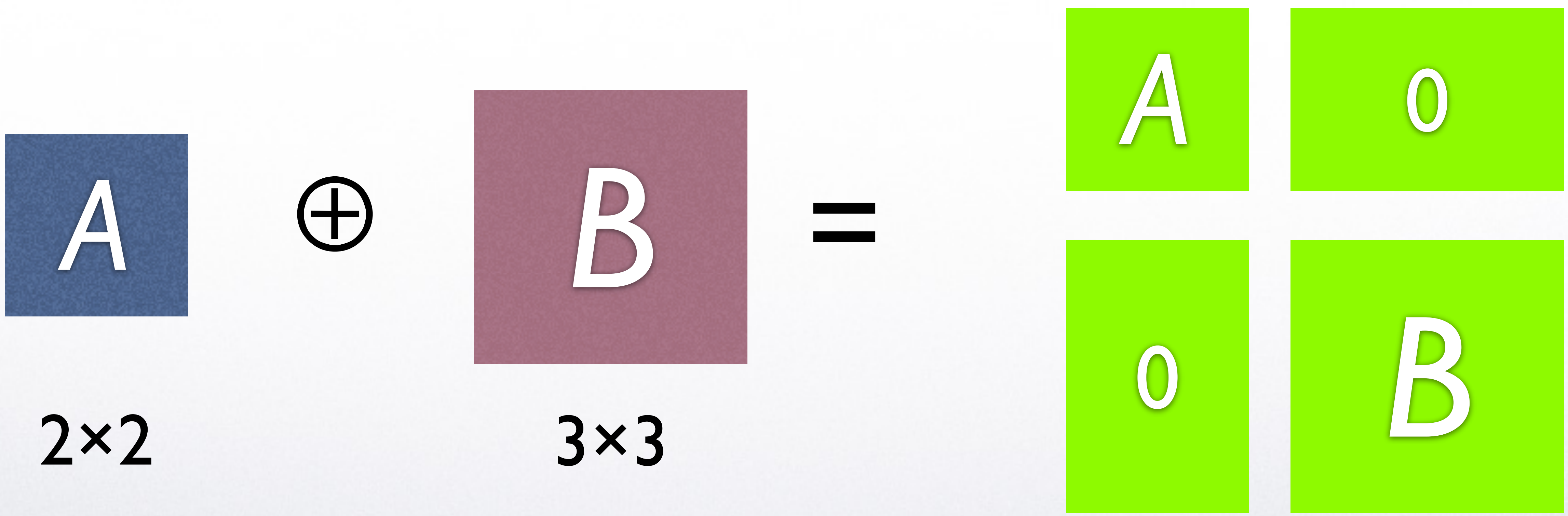

Tensor sum of tensors – 3D analog

5×5

Asymptotic sum inequality

Asymptotic sum inequality

Border rank

- Schönhage proved: $\underline{R}(\langle 4,1,4\rangle \oplus \langle 1,9,1\rangle) \leq 17$

Outer product Inner product

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} [y_1 y_2 y_3 y_4] \qquad [z_1 \ldots z_9] \begin{bmatrix} w_1 \\ \vdots \\ w_9 \end{bmatrix}$$

Asymptotic sum inequality

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  $\omega \leq 3\tau$ where $16^{\tau}+9^{\tau}=17$

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- Gives the bound $\omega < 2.55$
- Applies to any tensor sum of matrix multiplication tensors

Laser method

What about more general tensors?

Coppersmith–Winograd
"Easy identity"

$$\underline{R}\left(\sum_{i=1}^{q} x_0 y_i z_i + \sum_{i=1}^{q} x_i y_0 z_i + \sum_{i=1}^{q} x_i y_i z_0\right) \leq q+2$$

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Problem: tensors not disjoint!

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$\omega < 2.404$

Coppersmith–Winograd "Complicated identity"

$\underline{R}(T_{CW}) \leq q+2$, where

Coppersmith–Winograd tensor

$$T_{CW} = \sum_{i=1}^{q} x_0 y_i z_i + \sum_{i=1}^{q} x_i y_0 z_i + \sum_{i=1}^{q} x_i y_i z_0 + x_0 y_0 z_{q+1} + x_0 y_{q+1} z_0 + x_{q+1} y_0 z_0$$

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Basis of all algorithms since 1987!

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Recursive laser method

$\underline{R}(T_{CW}{}^{\otimes 2}) \leq (q+2)^2$, where

$T_{CW}{}^{\otimes 2}$ = sum of 15 non-disjoint tensors:
12 matrix multiplication tensors &
3 complicated tensors

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$$T_{112} = \underbrace{\sum_{i=1}^{q} x_{i0} y_{i0} z_{0(q+1)}}_{\langle 1,q,1 \rangle} + \underbrace{\sum_{i=1}^{q} x_{0i} y_{0i} z_{(q+1)0}}_{\langle 1,q,1 \rangle} + \underbrace{\sum_{i=1}^{q}\sum_{j=1}^{q} x_{i0} y_{0j} z_{ij}}_{\langle q,1,q \rangle} + \underbrace{\sum_{i=1}^{q}\sum_{j=1}^{q} x_{0j} y_{i0} z_{ij}}_{\langle q,1,q \rangle}$$

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Solution: generalized asymptotic sum inequality handles tensors analyzed by laser method

Recursive laser method

- Analyzing $T_{CW}^{\otimes 2}$:

Recursive laser method

- Analyzing $T_{CW}^{\otimes 2}$:
  - Analyze T_{112} using laser method

Recursive laser method

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  - Analyze $T_{CW}^{\otimes 2}$ using generalized laser method

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$\omega < 2.376$

Analyzing powers of T_{CW}

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C & W	1987	T_{CW}	$\omega < 2.3871900$
C & W	1987	$T_{CW}^{\otimes 2}$	$\omega < 2.3754770$
Stothers	2010	$T_{CW}^{\otimes 4}$	$\omega < 2.3729269$
Williams	2012	$T_{CW}^{\otimes 8}$	$\omega < 2.3728642$
Le Gall	2014	$T_{CW}^{\otimes 16}$	$\omega < 2.3728640$
Le Gall	2014	$T_{CW}^{\otimes 32}$	$\omega < 2.3728639$

Source: Le Gall 2014

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What is the limit of this approach?

Our work

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- Formalize recursive laser method
  Define $L(T^{\otimes k})$ = resulting bound on ω

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- Formalize recursive laser method
  Define $L(T^{\otimes k})$ = resulting bound on ω
- Formulate "laser method with merging"
  Define $L_M(T)$ = resulting bound on ω
- Prove that $L_M(T) \leq L(T^{\otimes k})$ for all k

Our work

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- Main theorem: lower bound on $L_M(T)$

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- Application 1: $L_M(T_{CW^{\otimes 16}}) \geq 2.3725$ (cf. $L(T_{CW^{\otimes 32}}) \approx 2.3728$)

Our work

- Main theorem: lower bound on $L_M(T)$
- Application I: $L_M(T_{CW^{\otimes 16}}) \geq 2.3725$ (cf. $L(T_{CW^{\otimes 32}}) \approx 2.3728$)
- Application II: $L_M(T_{CW}) \geq 2.3078$ (do not expect to be tight)

Laser method with merging

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Example: T_{CW}

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Not representative!

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  - *Weight*: quantity in asymptotic sum inequality
- Upper bound using volume argument (inspired by Cohn, Kleinberg, Szegedy, Umans)

The lower bound

Coppersmith & Winograd showed

$$L(T_S) \leq 3\log_q \frac{q+2}{2^{h(1/3)}}$$

Coppersmith–Winograd "simple" tensor

$$T_S = \sum_{i=1}^{q} x_0 y_i z_i + \sum_{i=1}^{q} x_i y_0 z_i + \sum_{i=1}^{q} x_i y_i z_0 \qquad \underline{R}(T_S) \leq q+2$$

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$$L(T_S) \leq 3\log_q \frac{q+2}{2^{h(1/3)}} \qquad q=8 \Rightarrow \omega<2.404$$

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Our lower bound is extension of their methods

Partitioned tensors

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x variables

x_0 $X^{[0]}$

$x_1,\ldots,x_q$ $X^{[1]}$

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x variables		*y* variables		*z* variables	
x_0	$X^{[0]}$	y_0	$Y^{[0]}$	z_0	$Z^{[0]}$
$x_1,\ldots,x_q$	$X^{[1]}$	$y_1,\ldots,y_q$	$Y^{[1]}$	$z_1,\ldots,z_q$	$Z^{[1]}$

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$$T_S = \underbrace{\sum_{i=1}^{q} x_0 y_i z_i}_{\langle 1,1,q\rangle^{[0,1,1]}} + \underbrace{\sum_{i=1}^{q} x_i y_0 z_i}_{\langle q,1,1\rangle^{[1,0,1]}} + \underbrace{\sum_{i=1}^{q} x_i y_i z_0}_{\langle 1,q,1\rangle^{[1,1,0]}}$$

x variables		y variables		z variables	
x_0	$X^{[0]}$	y_0	$Y^{[0]}$	z_0	$Z^{[0]}$
$x_1,\ldots,x_q$	$X^{[1]}$	$y_1,\ldots,y_q$	$Y^{[1]}$	$z_1,\ldots,z_q$	$Z^{[1]}$

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Can represent powers of T_S using *index triples*:

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$$T_S = [0,1,1]\ [1,0,1]\ [1,1,0]$$

$$T_S^{\otimes 2} = \begin{matrix} [00,11,11] & [01,10,11] & [01,11,10] \\ [10,01,11] & [11,00,11] & [11,01,10] \\ [10,11,01] & [11,10,01] & [11,11,00] \end{matrix}$$

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Zeroing variables: retain only index triples $[i,j,k]$ where $i \in I$, $j \in J$, $k \in K$

Partitioned tensors

Can represent powers of T_S using *index triples*:

$$T_S = [0,1,1]\ [1,0,1]\ [1,1,0]$$

$$T_S^{\otimes 2} = \begin{matrix} [00,11,11]\ [01,10,11]\ [01,11,10] \\ [10,01,11]\ [11,00,11]\ [11,01,10] \\ [10,11,01]\ [11,10,01]\ [11,11,00] \end{matrix}$$

Zeroing variables: retain only index triples $[i,j,k]$ where $i \in I$, $j \in J$, $k \in K$

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Partitioned tensors

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$I = \{00,11\}$ $J = \{01,11\}$ $K = \{10,11\}$

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Taking the limit $N \to \infty$: $\omega \leq 3\log_q \frac{q+2}{C}$, where $C = \lim_{N\to\infty} C_N{}^{1/N}$

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- It follows that $L(T_S) = \frac{q+2}{2^{h(1/3)}}$

More on merging

Coppersmith–Winograd tensor $T_{CW} = \sum_{i=1}^{q} x_0 y_i z_i + \sum_{i=1}^{q} x_i y_0 z_i + \sum_{i=1}^{q} x_i y_i z_0 + x_0 y_0 z_{q+1} + x_0 y_{q+1} z_0 + x_{q+1} y_0 z_0$

x variables		y variables		z variables	
x_0	$X[0]$	y_0	$Y[0]$	z_0	$Z[0]$
$x_1,\ldots,x_q$	$X[1]$	$y_1,\ldots,y_q$	$Y[1]$	$z_1,\ldots,z_q$	$Z[1]$
x_{q+1}	$X[2]$	y_{q+1}	$Y[2]$	z_{q+1}	$Z[2]$

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$$\underbrace{}_{\langle 1,1,q\rangle^{[0,1,1]}} \underbrace{}_{\langle q,1,1\rangle^{[0,1,1]}} \underbrace{}_{\langle 1,q,1\rangle^{[0,1,1]}} \underbrace{}_{\langle 1,1,1\rangle^{[0,0,2]}} \underbrace{}_{\langle 1,1,1\rangle^{[0,2,0]}} \underbrace{}_{\langle 1,1,1\rangle^{[2,0,0]}}$$

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More on merging

Actual merging in $T_{CW^{\otimes 2}}$:

$$x_0y_0z_{q+1}\otimes x_0y_{q+1}z_0+x_0y_{q+1}z_0\otimes x_0y_0z_{q+1}=$$

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Corresponds to merging [00,02,20] and [00,20,02]

In any merging, for each *t*: either

i_t = 0 for all [*i*,*j*,*k*] (*x-constant*) or

j_t = 0 for all [*i*,*j*,*k*] (*y-constant*) or

k_t = 0 for all [*i*,*j*,*k*] (*z-constant*)

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- Upper bound no. of index triples
- Deduce upper bound on capacity

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- Completely different methods?

Questions?