Triangle-intersecting families of graphs

David Ellis, **Yuval Filmus** and Ehud Friedgut

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At most $2^{\binom{n}{2}-3}$. In other words, triangle-juntas are optimal.

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Ellis, Filmus & Friedgut (2010)

Semidefinite method: $2^{\binom{n}{2}-3}$.

Shearer's lemma

Setup

- U: ground set.
- $\mathcal{F}, \mathcal{S} \subseteq 2^U$: families of subsets of U.
- $\mathcal{F}_S \subseteq 2^S$: projection of $\mathcal{F}$ into $S \subseteq U$.

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Shearer's lemma (1986)

If every $i \in U$ appears in at least $\mu|\mathcal{S}|$ sets of $\mathcal{S}$ then

$$|\mathcal{F}|^\mu \leq \sqrt[|\mathcal{S}|]{\prod_{S \in \mathcal{S}} |\mathcal{F}_S|}.$$

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Crucial property

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Method

Suppose A is a symmetric $2^{K_n} \times 2^{K_n}$ matrix such that:

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$$|\mathcal{F}| \leq \frac{1/7}{1+1/7} 2^{\binom{n}{2}} = \frac{1}{8} 2^{\binom{n}{2}}.$$

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Can assume $A(G, H) = A(G \oplus K, H \oplus K)$ for all G, H, K. ("A is *circulant*".)

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Conclusion

Eigenvectors of A are $\chi_K \colon G \mapsto (-1)^{|K \cap G|}$.

Admissible spectra

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$\lambda\colon 2^{K_n} \to \mathbb{R}$ is an *admissible spectrum* if for some A:

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Basis for admissible spectra

Admissible spectra are spanned by $\{G \mapsto (-1)^{|G \setminus B|} : \text{bipartite } B\}$.

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G	$q_0(G)$	$q_1(G)$	$q_2(G)$	$q_3(G)$	$q_4(G)$
$\varnothing$	1	0	0	0	0
$-$	1/2	1/2	0	0	0
$\wedge$	1/4	1/2	1/4	0	0
$\triangle$	1/4	0	3/4	0	0
$\wedge\wedge$	1/16	1/4	3/8	1/4	1/16
⟠	1/8	0	1/4	1/2	1/8

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$\diamond$	1/8	0	1/4	1/2	1/8

Observation

$\lambda_G = (-1)^{|G|} q_k(G)$ is admissible for all k.

The proof

Definition

$\lambda_G = (-1)^{|G|}\left(q_0(G) - \frac{5}{7}q_1(G) - \frac{1}{7}q_2(G) + \frac{3}{28}q_3(G)\right)$ admissible.

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- $\lambda_\varnothing = 1$ by inspection.

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- $\lambda_G \geq -1/7$ for large $|G|$ since $q_k(G) \approx 0$.
- $\lambda_G \geq -1/7$ for medium $|G|$ (boring calculations). □

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Other generalizations

Cross-intersecting families?

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Other generalizations

Cross-intersecting families?

Multiply-intersecting families?