Triangle-intersecting families of graphs

David Ellis, Yuval Filmus and Ehud Friedgut

5 December 2015
The problem

Definition

A family of subsets of $K_n$ is \textit{triangle-intersecting} if the intersection of any two graphs contains a triangle.

Example

Triangle-junta — all graphs containing a fixed triangle.

Question (Simonovits & Sós, 1976)

How many graphs can a triangle-intersecting family of graphs contain?

Conjecture (Simonovits & Sós, 1976)

At most $2^{n^2 - 3}$. In other words, triangle-juntas are optimal.
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Trivial upper bound: $2^{\binom{n}{2}} - 1$. 
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Shearer’s lemma

Setup

- $U$: ground set.
- $\mathcal{F}, S \subseteq 2^U$: families of subsets of $U$.
- $\mathcal{F}_S \subseteq 2^S$: projection of $\mathcal{F}$ into $S \subseteq U$.

Shearer’s lemma (1986)

If every $i \in U$ appears in at least $\mu|S|$ sets of $S$ then $|\mathcal{F}| \leq \mu \sum_{S \in S} |\mathcal{F}_S|$.
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If every $i \in U$ appears in at least $\mu |S|$ sets of $S$ then

$$|\mathcal{F}|^\mu \leq |S| \sqrt[|S|]{\prod_{S \in S} |\mathcal{F}_S|}.$$
Application to triangle-intersecting families

**Setup**

- $U = \text{edges of } K_n$. 

- Let $F$ be a triangle-intersecting family.
- Let $S$ be all complements of complete balanced bipartite graphs.

Then, $F_S$ is intersecting for all $S \in S$ implies $|F_S| \leq 2|S| - 1$.

Every edge appears in $\approx |S|/2$ sets.

Applying Shearer's lemma, $|F|_1/2 \leq |S| \sum_{F \in S} |F_S| \lesssim 2n^2/2 - 1 \implies |F| \leq 2n^2 - 2$. 

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Every edge appears in $\approx |S|/2$ sets.

Applying Shearer's lemma $|\mathcal{F}| \leq |S| \Omega(S) |\mathcal{F}_S| \approx 2^{n^2/2 - 1} \implies |\mathcal{F}| \leq 2^{n^2/2 - 2}$. 


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Applying Shearer's lemma $|\mathcal{F}| \leq \sum_{S \in S} |\mathcal{F}_S| \leq 2^{n/2} - 2$. 

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**Applying Shearer's lemma**

$$|\mathcal{F}|^{1/2} \leq |S| \sqrt{\prod_{S \in S} |\mathcal{F}_S|}$$
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Applying Shearer's lemma

\[
|\mathcal{F}|^{1/2} \leq |S| \sqrt{\prod_{S \in S} |\mathcal{F}_S|} \leq 2^{\left(\begin{array}{c} n \\ 2 \end{array}\right)/2-1}
\]
Application to triangle-intersecting families

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- $U =$ edges of $K_n$.
- $\mathcal{F}$: triangle-intersecting family.
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- Every edge appears in $\approx |S|/2$ sets.

Applying Shearer's lemma

$$|\mathcal{F}|^{1/2} \leq |S| \prod_{S \in S} |\mathcal{F}_S| \lesssim 2^{n/2-1} \implies |\mathcal{F}| \leq 2^{n/2}.$$
Heart of the argument

**Crucial property**

\( \mathcal{F} \) triangle-intersecting, \( \mathcal{B} \) bipartite \( \implies \) \( \mathcal{F}_\overline{\mathcal{B}} \) intersecting.
Heart of the argument

**Crucial property**
\[ \mathcal{F} \text{ triangle-intersecting, } \mathcal{B} \text{ bipartite } \implies \mathcal{F}_{\overline{\mathcal{B}}} \text{ intersecting.} \]

**Observation**
\[ \mathcal{F} \text{ odd-cycle-intersecting, } \mathcal{B} \text{ bipartite } \implies \mathcal{F}_{\overline{\mathcal{B}}} \text{ intersecting.} \]
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\[ \mathcal{F} \text{ triangle-intersecting, } \mathcal{B} \text{ bipartite } \Longrightarrow \mathcal{F}_B \text{ intersecting.} \]

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\[ \mathcal{F} \text{ odd-cycle-intersecting, } \mathcal{B} \text{ bipartite } \Longrightarrow \mathcal{F}_B \text{ intersecting.} \]

Conclusion

\[ \mathcal{F} \text{ odd-cycle-intersecting } \Longrightarrow |\mathcal{F}| \leq 2^{\binom{n}{2} - 2}. \]
Agreeing families

Agreement

\[ \mathcal{F} \subseteq 2^{K_n} \iff \text{family of 2-colorings of } K_n. \]
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\[ \mathcal{F} \text{ odd-cycle-agreeing, } B \text{ bipartite} \implies \overline{\mathcal{F}_B} \text{ agreeing.} \]
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\[ F \subseteq 2^{K_n} \iff \text{family of 2-colorings of } K_n. \]

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\[ F \text{ odd-cycle-agreeing, } B \text{ bipartite } \implies F_{\overline{B}} \text{ agreeing.} \]

Agreeing families contain \( \leq \frac{1}{2} \) of the sets \( \implies |F_{\overline{B}}| \leq 2^{|\overline{B}|-1}. \)
## Agreeing families

### Agreement

$\mathcal{F} \subseteq 2^{K_n} \iff$ family of 2-colorings of $K_n$.

**Odd-cycle-agreeing family**: every two graphs agree on colors of some odd cycle.

### Observation

$\mathcal{F}$ odd-cycle-agreeing, $\mathcal{B}$ bipartite $\implies \overline{\mathcal{F}_\mathcal{B}}$ agreeing.

Agreeing families contain $\leq \frac{1}{2}$ of the sets $\implies |\overline{\mathcal{F}_\mathcal{B}}| \leq 2^{\overline{\mathcal{B}}-1}$.

### Conclusion

$\mathcal{F}$ odd-cycle-agreeing $\implies |\mathcal{F}| \leq 2^{n\choose 2} - 2$. 

[Note: The notation $\frac{1}{2}$ and $2^{n\choose 2} - 2$ are used to represent bounds on the size of certain sets in the context of the agreement families.]
Suppose $A$ is a symmetric $2^{K_n} \times 2^{K_n}$ matrix such that:

- $A(G, H) = 0$ if $G, H$ don’t agree on some odd cycle.
Method

Suppose $A$ is a symmetric $2^{Kn} \times 2^{Kn}$ matrix such that:

- $A(G, H) = 0$ if $G, H$ don’t agree on some odd cycle.
- $A1 = 1$. 

Proof

We construct a matrix with $\lambda = -\frac{1}{7}$. 

$|F| \leq \frac{1}{7} + \frac{1}{7^2}n^2 = \frac{82}{7^2}n^2.$
Suppose $A$ is a symmetric $2^{K_n} \times 2^{K_n}$ matrix such that:

- $A(G, H) = 0$ if $G, H$ don’t agree on some odd cycle.
- $A\mathbf{1} = \mathbf{1}$.
- Minimal eigenvalue of $A$ is $\lambda$. 

Then for all odd-cycle-agreeing families $F$:

$$|F| \leq -\lambda 1 - \lambda 2 n^2.$$ 

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$$|\mathcal{F}| \leq \frac{-\lambda}{1 - \lambda} 2^{\binom{n}{2}}.$$
Semidefinite method

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$$|\mathcal{F}| \leq \frac{1/7}{1 + 1/7}2^{\binom{n}{2}} = \frac{1}{8}2^{\binom{n}{2}}.$$
Symmetry considerations

Observation
If $\mathcal{F}$ is odd-cycle-agreeing then for all sets of edges $K$, 
$\{G \oplus K : G \in \mathcal{F}\}$ is odd-cycle-agreeing.
Symmetry considerations

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**Corollary**

Can assume $A(G, H) = A(G \oplus K, H \oplus K)$ for all $G, H, K$. 
(“$A$ is circulant”.)
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**Conclusion**
Eigenvectors of $A$ are $\chi_K : G \mapsto (-1)^{|K \cap G|}$. 
Admissible spectra

**Definition**

\[ \lambda : 2^{K_n} \rightarrow \mathbb{R} \] is an *admissible spectrum* if for some \( A \):

- \( A(G, H) = 0 \) if \( G, H \) don’t agree on some odd cycle.
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Find an admissible spectrum $\lambda$ with:

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**Basis for admissible spectra**

Admissible spectra are spanned by \( \{ G \mapsto (-1)^{|G\setminus B|} : \text{bipartite } B \} \).
Cut statistics

**Definition**

\( q_k(G) = \) probability that a random partition of \( G \) cuts \( k \) edges.
Definition

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### Observation

\[ \lambda_G = (-1)^{|G|} q_k(G) \text{ is admissible for all } k. \]
The proof

**Definition**

\[ \lambda_G = (-1)^{|G|} \left( q_0(G) - \frac{5}{7} q_1(G) - \frac{1}{7} q_2(G) + \frac{3}{28} q_3(G) \right) \text{ admissible.} \]
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**Claim**

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- \( \lambda_G \geq -1/7 \) for \( G \) in table.
The proof

**Definition**

\[ \lambda_G = (-1)^{|G|} \left( q_0(G) - \frac{5}{7} q_1(G) - \frac{1}{7} q_2(G) + \frac{3}{28} q_3(G) \right) \text{ admissible.} \]

**Table**

**Claim**

\[ \lambda_\emptyset = 1 \text{ and } \lambda_G \geq -1/7 \text{ for all } G. \]

**Proof**

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- \( \lambda_G \geq -1/7 \) for medium \( |G| \) (boring calculations).
Uniqueness and stability

Upper bound

$\mathcal{F}$ odd-cycle-agreeing $\implies |\mathcal{F}| \leq 2^{\binom{n}{2} - 3}$. 
Uniqueness and stability

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**Stability**

\[ \mathcal{F} \text{ odd-cycle-agreeing, } |\mathcal{F}| \approx 2^{\binom{n}{2} - 3} \implies \mathcal{F} \approx \text{ a triangle-junta}. \]
Open questions

Other graphs

What happens if we replace *triangle* with other graphs?
Open questions

Other graphs
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Path of length 3
Christsofides: can beat $2^{\binom{n}{2}} - 3$ for $P_3$!
Open questions

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Other generalizations
Cross-intersecting families?
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Other generalizations
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