Analysis of Boolean functions on *exotic* domains

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PART I
What is analysis of Boolean functions?

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(mostly)

the study of Boolean-valued functions

from a spectral perspective.

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The functions are usually over a finite domain.

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- Classification function (statistical learning theory)

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- More rarely, over other product domains
- Some applications: over finite groups or over $\binom{[n]}{k}$

Known as the "slice" or the Johnson association scheme

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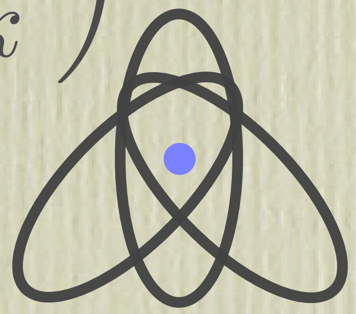

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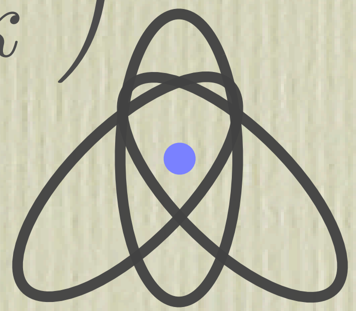

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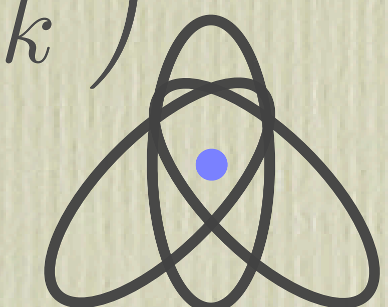

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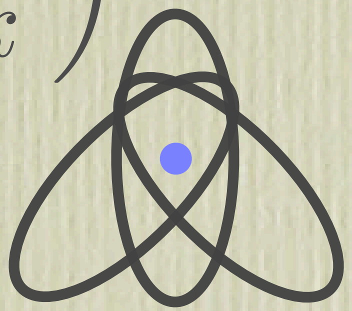

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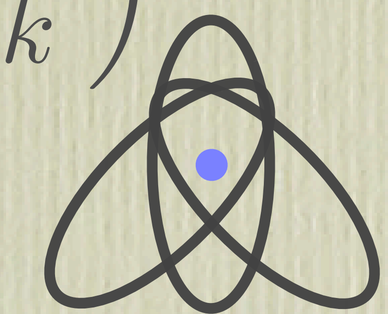

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  - Only known proof through Analysis

PART II
Structure theorems

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Implies stability for Erdős–Ko–Rado

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- Ellis–F.–Friedgut (2):
  Every *balanced* Boolean function close to a linear function is close to a dictatorship

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- Ellis–F.–Friedgut (3): Same true for S_n for sparse functions

PART III
Invariance principle

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- Berry–Esséen: as long as no α_i is too prominent

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- Properties of Gaussians imply

$$\sum_i \alpha_i g_i \sim N(0, \sum_i \alpha_i^2)$$

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- Important corollaries in theoretical computer science

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- For example, $x_1^2+\ldots+x_n^2$ is constant while $g_1^2+\ldots+g_n^2$ is not
- Not a problem, since any function on Boolean cube has unique multilinear representation ("Fourier expansion")

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Middle slice is a representative section of the Boolean cube from the point of view of low-degree "harmonic" multilinear polynomials

PART IV
Gelfand–Tsetlin basis for the slice

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- On S_n we have representation theory
- On the slice we also have representation theory: every multilinear “harmonic” polynomial decomposes as sum of its homogeneous parts (part of general theory of association schemes)

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- Basis depends on order of coordinates

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- Norm of χ_B proportional to norm of

$$(x_1 - x_2) \cdots (x_{2|B|-1} - x_{2|B|})$$

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- Basis is explicit orthogonal basis of eigenvectors for Johnson and Kneser graphs
- Simplifies Wimmer's proof of Friedgut's junta theorem on the slice

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- More theorems on slice and symmetric group

THE END