Analysis of Boolean functions on exotic domains

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PART I
What is analysis of Boolean functions?
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(mostly) \{0,1\} or \{±1\}

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Fourier analysis, representation theory, or equivalent from a spectral perspective.

The functions are usually over a finite domain.
Where do Boolean functions come from?
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- Collection of subsets of a finite set (extremal combinatorics)
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- Subset of vertices in a graph (theoretical computer science)
- Classification function (statistical learning theory)
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- Some applications: over finite groups or over $\binom{[n]}{k}$.

Known as the “slice” or the Johnson association scheme.
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• Analysis: almost extremal families close to stars

Every two sets intersect
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“Stability”
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• Only known proof through Analysis
PART II
Structure theorems
• If a Boolean function on \( \{0, 1\}^n \) satisfies

\[
f(x_1, \ldots, x_n) = C + \sum_{i=1}^{n} a_i x_i
\]

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Implies stability for Erdős–Ko–Rado
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Dictatorship on $S_n$
Friedgut–Kalai–Naor on $S_n$
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- $\max(x_{11}, x_{22})$ close to $x_{11} + x_{22}$ but not to a dictatorship
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- Ellis–F–Friedgut ($\tau$):
  Every Boolean function of magnitude $c/n$ (for $c$ small)
  which is close to a linear function is close to a maximum of $c$ entries
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  Every Boolean function of magnitude $c/n$ (for $c$ small) which is close to a linear function is close to a maximum of $c$ entries

- Ellis–F.–Friedgut (2):
  Every balanced Boolean function close to a linear function is close to a dictatorship
Higher-degree analogues
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• Nisan–Szegedy: If a Boolean function equals a degree $d$ polynomial then it is a $C(d)$-“junta” (depends on $C(d)$ coordinates)
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  • Stability for $t$-intersecting families of sets

• Ellis–F.–Friedgut (3): Same true for $S_n$ for sparse functions
PART III
Invariance principle
Central limit theorem
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- Suppose $x_i$ are random $\pm 1$ variables
Central limit theorem

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• Under what conditions does

$$\sum_{i} \alpha_i x_i$$

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Central limit theorem

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• Berry–Esséen: as long as no $\alpha_i$ is too prominent
Lindeberg replacement trick
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• How to prove Berry–Esséen?
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- Replace each $x_i$ by standard Gaussian $g_i$: 
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• Replace each $x_i$ by standard Gaussian $g_i$:

$$\sum_i \alpha_i x_i \approx \sum_i \alpha_i g_i$$
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• How to prove Berry–Esséen?
• Replace each $x_i$ by standard Gaussian $g_i$:

$$\sum_i \alpha_i x_i \approx \sum_i \alpha_i g_i$$

• Properties of Gaussians imply

$$\sum_i \alpha_i g_i \sim N(0, \sum_i \alpha_i^2)$$
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- Implies that Majority vote is the voting rule most resistant to noise (asymptotically).
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- This time we need all the variable influences to be small.
- Implies that Majority vote is the voting rule most resistant to noise (asymptotically).
- Important corollaries in theoretical computer science.
Caveat
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- Invariance principle only applies to (low-degree) multilinear polynomials.
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- For example, $x_1^2 + \ldots + x_n^2$ is constant while $g_1^2 + \ldots + g_n^2$ is not
Invariance principle only applies to (low-degree) multilinear polynomials

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Not a problem, since any function on Boolean cube has unique multilinear representation ("Fourier expansion")
Invariance principle on the slice
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- Relevant distributions:
Invariance principle on the slice

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  - \((x_1,\ldots,x_n)\): product distribution with same marginals (\(\text{Ber}(k/n)\))
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  - \((x_1, \ldots, x_n)\): product distribution with same marginals \((\text{Ber}(k/n))\)
  - \((g_1, \ldots, g_n)\): Gaussian product distribution with same mean and variance
  - \(s_1 + \ldots + s_n\) constant but \(x_1 + \ldots + x_n\) isn’t!
Invariance principle on the slice
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• Solution (Dunkl): Consider only multilinear polynomials satisfying:

\[ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} = 0 \]
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• Corollary: Majority is Stablest on the slice
Invariance principle on the slice

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  - Every function on slice has a unique multilinear representation of this form
  - F. Kindler–Mossel–Wimmer: invariance principle holds for such polynomials

Corollary: Majority is Stablest on the slice

\[ \sum_{x \in \{-1,1\}^n} f(x) = \sum_{x \in \{-1,1\}^n \mid \sum x_i = 0} f(x) \]

Middle slice is a representative section of the Boolean cube from the point of view of low-degree “harmonic” multilinear polynomials

Invariance principle on the slice
PART IV
Gelfand–Tsetlin basis for the slice
Fourier analysis on the slice
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- On the Boolean cube we have Fourier analysis on $\mathbb{Z}_2^n$
Fourier analysis on the slice

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- On $S_n$ we have representation theory
Fourier analysis on the slice

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• On $S_n$ we have representation theory

• On the slice we also have representation theory: every multilinear “harmonic” polynomial decomposes as sum of its homogeneous parts (part of general theory of association schemes)
Gelfand–Tsetlin basis
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- Srinivasan constructs (implicitly) a canonical orthogonal basis for functions on the slice
Gelfand–Tsetlin basis

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- Basis is orthogonal with respect to all symmetric measures on $x_1, \ldots, x_n$!
Gelfand–Tsetlin basis

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- Basis is orthogonal with respect to all symmetric measures on $x_1, \ldots, x_n$
- Basis depends on order of coordinates
Explicit basis
Explicit basis

• F. gives explicit construction of the basis
Explicit basis

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- Definition: $A = (a_1, \ldots, a_\ell) < B = (b_1, \ldots, b_\ell)$ if
  $$a_1 < b_1, \ldots, a_\ell < b_\ell, b_1 < \cdots < b_\ell$$
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- Basis consists of all non-zero functions

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\chi_B = \sum_{A < B} (x_{a_1} - x_{b_1}) \cdots (x_{a_\ell} - x_{b_\ell})
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- Norm of \( \chi_B \) proportional to norm of

  \[
  (x_1 - x_2) \cdots (x_2|B|^1_1 - x_2|B|)
  \]
Implications
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- For small $d$, norm of $(x_1 - x_2) \cdots (x_{2d-1} - x_{2d})$ roughly identical for uniform distribution on $k$-slice and corresponding product distribution.
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- Basis is explicit orthogonal basis of eigenvectors for Johnson and Kneser graphs
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- Simplifies Wimmer’s proof of Friedgut’s junta theorem on the slice
OPEN QUESTIONS
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• More exotic domains:
OPEN QUESTIONS

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  • $k$-dimensional vector subspaces of $n$-dimensional vector space over finite field
OPEN QUESTIONS

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  • “Multi-slice”
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• More theorems on slice and symmetric group
THE END