# AND testing

Yuval Filmus, Noam Lifshitz, Dor Minzer, Elchanan Mossell

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**Abstract**

If a function $f\colon \{0,1\}^n \to \{0,1\}$ satisfies $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in \{0,1\}^n$ then either $f = 0$ or $f$ is an AND of a subset of coordinates. We show that if $f(x \wedge y) = f(x) \wedge f(y)$ holds with probability $1 - \epsilon$ then $f$ is $\delta$-close to 0 or to an AND, where $\delta = 1/\log^{\Omega(1)}(1/\epsilon)$. This improves on a result of Nehama (in which $\delta$ depends on $n$) and substantially simplifies a result by the authors.

## 1 Introduction

Which functions $f\colon \{0,1\}^n \to \{0,1\}$ satisfy $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in \{0,1\}^n$ (where $x \wedge y$ is bitwise AND)? It is not hard to show that the set of solutions includes $f = 0$ and $f = \bigwedge_{i \in S} x_i$ (when $S = \emptyset$, this is just $f = 1$). In this note, our goal is to prove a stability version of this result:

**Theorem 1.** *Suppose that $f\colon \{0,1\}^n \to \{0,1\}$ satisfies*

$$\Pr_{x,y \in \{0,1\}^n}[f(x \wedge y) = f(x) \wedge f(y)] = 1 - \epsilon,$$

*where $x, y$ are chosen uniformly at random from $\{0,1\}^n$.*

*Then $f$ is $\delta$-close to 0 or to an AND of a subset of the coordinates, where $\delta = 1/\log^{\Omega(1)}(1/\epsilon)$.*

We conjecture that Theorem 1 holds with $\delta = O(\epsilon)$.

Theorem 1 improves on a result of Nehama [Neh13], in which $\delta$ depends on $n$ (but has much better dependence on $\epsilon$). Theorem 1 also appears in our work [FLMM20], but the proof here is substantially simpler.

**Preliminaries** We assume that the reader is familiar with Boolean function analysis. The only deep result we use is Bourgain's theorem, in a form due to Kindler, Kirshner and O'Donnell [KKO18]:

**Theorem 2** ([KKO18, Theorem 1.6])**.** *If $f\colon \{0,1\}^n \to \{0,1\}$ satisfies $\|f^{>k}\|^2 \le \epsilon$ then $f$ is $O(\epsilon\sqrt{k})$-close to a Boolean junta depending on $\epsilon^{-4} 2^{O(k)}$ variables.*

(The original version refers to functions from $\{\pm 1\}^n$ to $\{\pm 1\}$, but this only affects some constants.)

## 2 Proof

Let $f\colon \{0,1\}^n \to \{0,1\}$ be a function that satisfies

$$\Pr_{x,y}[f(xy) = f(x)f(y)] = 1 - \epsilon.$$

(We replaced $\wedge$ with product since they are the same for $\{0,1\}$ variables.)

The general idea of the proof is to show that $f$ is close to a junta, using Theorem 2. The first step is to relate $f$ to a noisy version of itself.

**Lemma 3.** *Let* $\mu := \mathbb{E}[f]$ *and*

$$T_{\downarrow}f(x) = \mathop{\mathbb{E}}_{y}[f(xy)].$$

*We have*

$$\|T_{\downarrow}f - \mu f\|^2 = O(\epsilon).$$

*Proof.* This follows from

$$\|T_{\downarrow}f - \mu f\|^2 = \mathop{\mathbb{E}}_{x}\left[\left(\mathop{\mathbb{E}}_{y}[f(xy)] - \mathop{\mathbb{E}}_{y}[f(x)f(y)]\right)^2\right] \leq \mathop{\mathbb{E}}_{x,y}[(f(xy) - f(x)f(y))^2] = O(\epsilon). \qquad \square$$

This lemma is helpful since $T_{\downarrow}f$ has small Fourier tails. To show this, we give a formula for the Fourier expansion of $T_{\downarrow}f$ in terms of the *biased* Fourier expansion of $f$.

**Lemma 4.** *Let the* $1/4$*-biased Fourier expansion of* $f$ *be*

$$f = \sum_{S \subseteq [n]} c_S \omega_S, \text{ where } \omega_S = \prod_{i \in S} \frac{4x_i - 1}{\sqrt{3}}.$$

*(The functions* $\omega_S$ *form an orthonormal basis with respect to* $\mu_{1/4}$*.) Then*

$$\hat{f}(S) = \left(\frac{1}{\sqrt{3}}\right)^{|S|} c_S.$$

*(This is the coefficient of the Fourier character* $\prod_{i \in S}(2x_i - 1)$*.)*

*Proof.* It suffices to consider $T_{\downarrow}\omega$, where $\omega = (4x-1)/\sqrt{3}$. Direct computation gives

$$T_{\downarrow}\omega = \frac{1}{2} \cdot \frac{4x-1}{\sqrt{3}} + \frac{1}{2} \cdot \frac{-1}{\sqrt{3}} = \frac{2x-1}{\sqrt{3}}. \qquad \square$$

We can now bound the Fourier tails of $f$.

**Lemma 5.** *For every* $k$*,*

$$\|f^{>k}\|^2 \leq O\big(\sqrt{1/3}^k + \epsilon\big)\mu^{-2}.$$

*Proof.* Lemma 4 shows that

$$\|(T_{\downarrow}f)^{>k}\|^2 \leq \left(\frac{1}{\sqrt{3}}\right)^k \sum_{|S|>k} c_S^2 \leq \left(\frac{1}{\sqrt{3}}\right)^k,$$

since

$$\sum_S c_S^2 = \mathop{\mathbb{E}}_{\mu_{1/4}}[f^2] \leq 1.$$

Lemma 3 now implies that

$$\mu^2\|f^{>k}\|^2 = \|(\mu f)^{>k}\|^2 \leq 2\|(T_{\downarrow}f)^{>k}\|^2 + 2\|(T_{\downarrow}f)^{>k} - (\mu f)^{>k}\|^2 \leq$$
$$O((1/\sqrt{3})^k) + 2\|T_{\downarrow}f - \mu f\|^2 = O((1/\sqrt{3})^k + \epsilon). \qquad \square$$

Invoking Theorem 2, we approximate $f$ by a junta.

**Lemma 6.** *Suppose that* $\mu \geq \sqrt[3]{\epsilon}$*. There exists a constant* $K > 0$ *such that for every* $N < (1/\epsilon)^K$ *there is a junta* $F$*, depending on* $N$ *coordinates, such that*

$$\|f - F\|^2 = \frac{1}{N^{\Omega(1)}}.$$

*Proof.* Let $k$ be such that $\sqrt{1/3}^k \geq \epsilon$. Lemma 5 shows that $\|f^{>k}\|^2 = O(\sqrt{1/3}^k \sqrt[3]{\epsilon}) = O(\rho^k)$, for some $\rho < 1$. According to Theorem 2, $f$ is $O(\rho^k \sqrt{k})$-close to a junta depending on $2^{O(k)}$ variables. Choosing $k = c \log N$ for an appropriate constant $c > 0$ completes the proof. ☐

Let us now see how this helps us.

**Lemma 7.** *Suppose that $\mu \geq \sqrt[3]{\epsilon}$. Let $N < (1/\epsilon)^K$, and let $F$ be the junta promised by Lemma 6. For an assignment $\alpha$ to the non-junta variables, let $f_\alpha$ be the corresponding restriction of $f$.*

*There exist assignments $\alpha, \beta$ to the non-junta variables such that*

$$\Pr_{x,y}[f_\alpha(x) f_\beta(y) = f_{\alpha\beta}(xy)] \geq 1 - \kappa\epsilon$$

*for some constant $\kappa > 0$, and $f_\alpha, f_\beta$ are $N^{-\Omega(1)}$ close to $F$.*

*Proof.* If we choose $\alpha, \beta$ at random then

$$\mathop{\mathbb{E}}_{\alpha,\beta}\big[\Pr_{x,y}[f_\alpha(x) f_\beta(y) \neq f_{\alpha\beta}(xy)]\big] = \epsilon.$$

Furthermore,

$$\mathop{\mathbb{E}}_{\alpha}\big[\Pr[F \neq f_\alpha]\big] = \Pr[F \neq f] = \frac{1}{N^{\Omega(1)}}$$

according to Lemma 6, and similarly for $f_\beta$. In particular, the probability (over $\alpha, \beta$) that each of these probabilities exceeds its expectation by a factor of 4 is at most 1/4. Applying the union bound, there is a choice of $\alpha, \beta$ for which none of these probabilities exceeds its expectation by more than a factor of 4. ☐

If we choose $N$ small enough, then the identity $f_\alpha(x) f_\beta(y) = f_{\alpha\beta}(xy)$ will *always* hold. In this case, the solutions are given by the following easy lemma.

**Lemma 8.** *Suppose that $a, b, c \colon \{0,1\}^m \to \{0,1\}$ satisfy*

$$a(x)b(y) = c(xy)$$

*for all $x, y \in \{0,1\}^m$. Then either $a = c = 0$, or $b = c = 0$, or there exists a set $S \subseteq [m]$ such that*

$$a(x) = b(x) = c(x) = \prod_{i \in S} x_i.$$

*Proof.* If $a = 0$ or $b = 0$ then clearly $c = 0$, so we can assume that $a, b \neq 0$.

We expand $a, b, c$ in terms of the basis $x_S := \prod_{i \in S} x_i$:

$$a(x) = \sum_{S \subseteq [m]} \tilde{a}(S) x_S,$$

and similarly for $b, c$. The condition $a(x)b(y) = c(xy)$ translates to

$$\sum_{S,T \subseteq [m]} \tilde{a}(S)\tilde{b}(T) x_S y_T = \sum_{S \subseteq [m]} \tilde{c}(S) x_S y_S.$$

Comparing coefficients (using the fact that $x_S y_T$ is a basis for all functions on $x, y$), we see that if $\tilde{a}(S) \neq 0$ then $\tilde{b}(T) = 0$ for all $T \neq S$.

Since $a \neq 0$, there must be some $S$ such that $\tilde{a}(S) \neq 0$. If $\tilde{a}(S') \neq 0$ for some $S' \neq S$ then $b = 0$, contradicting our assumption. Thus $a$ and $b$ are both multiples of $x_S$. Since $a$ and $b$ are Boolean, necessarily $a = b = x_S$, and so $c = x_S$ as well. ☐

We can now put everything together, proving Theorem 1.

*Proof of Theorem 1.* If $\mu < \sqrt[3]{\epsilon}$ then $f$ is $\sqrt[3]{\epsilon}$-close to 0, so we can assume that $\mu \geq \sqrt[3]{\epsilon}$.

We apply Lemma 7 with $N = \log(1/\kappa\epsilon) - 1$, so that $2^{-N} < \kappa\epsilon$; note that $N < (1/\epsilon)^K$ unless $\epsilon$ is larger than some constant, in which case the theorem trivializes. It follows that $f_\alpha(x) f_\beta(y) = f_{\alpha\beta}(xy)$ holds for all inputs $x, y$, and so $f_\alpha$ is either 0 or an AND, by Lemma 8. The theorem follows since $F$ is $1/N^{\Omega(1)}$-close to $f_\alpha$ (Lemma 7) and $f$ is $1/N^{\Omega(1)}$-close to $F$ (Lemma 6). ☐

## References


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[Neh13] Ilan Nehama. Approximately classic judgement aggregation. *Ann. Math. Artif. Intell.*, 68(1-3):91–134, 2013.