

A comment on *Intersecting Families of Permutations*

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Abstract

Ellis, Friedgut and Pilpel [EFP11] prove that for large enough n , a t -intersecting family of permutations contains at most $(n-t)!$ permutations. Their main theorem also states that equality holds only for t -cosets. We show that their proof of the characterization of extremal families is wrong. However, the characterization follows from a paper of Ellis [Ell11], as mentioned already by Ellis, Friedgut and Pilpel.

1 Introduction

The classical Erdős–Ko–Rado theorem states that when $n > 2k$, an intersecting family of k -subsets of $[n] := \{1, \dots, n\}$ contains at most $\binom{n-1}{k-1}$ sets, and moreover this is achieved only by *stars*, consisting of all sets containing a specific point. Wilson [Wil84] extended this result to t -intersecting families: he showed that when $n > (t+1)(k-t+1)$, a t -intersecting family of k -subsets of $[n]$ contains at most $\binom{n-t}{k-t}$ sets, and moreover this is achieved only by t -stars, consisting of all sets containing t specific points.

The Erdős–Ko–Rado theorem has been extended in many different directions, one of them to other domains. One of the most intriguing domains is that of *permutations*. We say that two permutations $\pi, \sigma \in S_n$ are t -intersecting if there exist t points $i_1 < \dots < i_t$ such that $\pi(i_1) = \sigma(i_1), \dots, \pi(i_t) = \sigma(i_t)$. A t -intersecting family of permutations is one in which every two permutations t -intersect.

A simple partitioning argument shows that any 1-intersecting family of permutations contains at most $(n-1)!$ permutations. This bound is only achieved by *cosets* $T_{ij} = \{\pi \in S_n : \pi(i) = j\}$, though this is surprisingly hard to show (different proofs appear in [CK03, LM04, GM09, EFP11]); see also the corresponding stability result of Ellis [Ell12a].

In a groundbreaking paper, Ellis, Friedgut and Pilpel [EFP11] showed that for any t there exists C_t such that for all $n \geq C_t$, any t -intersecting family of permutations contains at most $(n-t)!$ permutations, and furthermore this is only achieved by t -cosets $T_{\alpha_1 \dots \alpha_t \mapsto \beta_1 \dots \beta_t} = \{\pi \in S_n : \pi(\alpha_1) = \beta_1, \dots, \pi(\alpha_t) = \beta_t\}$.

Unfortunately, the proof in [EFP11] that t -cosets are the only t -intersecting families of permutations of maximum size is wrong for $t > 1$, as we indicate below. Fortunately, the main result of Ellis [Ell11] implies that this theorem is correct; this is mentioned in [EFP11] as an alternative proof of the characterization of t -intersecting families of maximum size. The same problem affects the characterization of the optimal families for setwise- t -intersecting families of permutations in Ellis [Ell12b], but according to Ellis (private communication), the result can be recovered using methods similar to those in [Ell11]. Yet another affected paper is Wang [Wan13], whose characterization of the optimal t -intersecting families in B_n and D_n relies on the affected parts of [EFP11].

The remainder of this note is structured as follows. In Section 2 we outline the relevant portion of the arguments of [EFP11]. In Section 3 and Section 4 we present counterexamples to two results in [EFP11]. We close the note with some final remarks in Section 5.

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2 The argument of Ellis–Friedgut–Pilpel

Here is the main theorem of [EFP11].

Theorem 3 *For any $k \in \mathbb{N}$, and any n sufficiently large depending on k , if $I \subset S_n$ is k -intersecting, then $|I| \leq (n - k)!$. Equality holds if and only if I is a k -coset of S_n .*

This first step toward proving Theorem 3 is the following result, which combines parts of Theorem 5 and Theorem 6 of [EFP11].

Theorem 5/6 *For any $k \in \mathbb{N}$ and any n sufficiently large depending on k , if $I \subset S_n$ is k -intersecting, then $|I| \leq (n - k)!$. Moreover, if $I \subset S_n$ is a k -intersecting family of size $(n - k)!$, then $1_I \in V_k$.*

Here 1_I is the characteristic function of the family I , and V_k is the linear span of the characteristic functions of the k -cosets. Given Theorem 5/6, to complete the proof of Theorem 3 we need to show that if I is a k -intersecting family of size $(n - k)!$, then I is a k -coset. The authors deduce this from the following result (called Theorem 8 in the introduction of [EFP11]), in which \mathcal{A}_k consists of all ordered k -tuples of distinct numbers in $[n]$.

Theorem 27 *Let $f \in V_k$ be nonnegative. Then there exist nonnegative coefficients $(b_{\alpha,\beta})_{\alpha,\beta \in \mathcal{A}_k}$ such that $f = \sum b_{\alpha,\beta} 1_{T_{\alpha \rightarrow \beta}}$. Furthermore, if f is Boolean, then f is the characteristic function of a disjoint union of k -cosets.*

Unfortunately, this theorem is wrong, as we show in Section 3. However, the special case $k = 1$, proved in [EFP11] separately as Theorem 28, is correct.

The proof of Theorem 27 relies on two theorems, Theorem 29 and Theorem 30. Of these, Theorem 29 is incorrect, as we show in Section 4, in which we also indicate the mistake in the proof.

3 Counterexample to Theorem 27

From now on we denote the characteristic function of a set S by $\llbracket S \rrbracket$ instead of 1_S , to increase legibility.

Let $n \geq 6$, and consider the following function:

$$f(\pi) = \llbracket \pi(\{1, 2, 3\}) = \{1, 2, 3\} \text{ or } \pi(\{1, 2, 3\}) \cap \{1, 2, 3\} = \emptyset \rrbracket.$$

This is clearly a Boolean function. We claim that it belongs to V_2 . Indeed, we can rewrite it in the following form, which makes it clear that it belongs to V_2 :

$$\begin{aligned} f(\pi) = & 1 - \llbracket \pi(1) \in \{1, 2, 3\} \rrbracket - \llbracket \pi(2) \in \{1, 2, 3\} \rrbracket - \llbracket \pi(3) \in \{1, 2, 3\} \rrbracket \\ & + \llbracket \pi(\{1, 2\}) \subset \{1, 2, 3\} \rrbracket + \llbracket \pi(\{1, 3\}) \subset \{1, 2, 3\} \rrbracket + \llbracket \pi(\{2, 3\}) \subset \{1, 2, 3\} \rrbracket. \end{aligned}$$

If Theorem 27 were true, then the support of f would be the disjoint union of 2-cosets. In particular, it would contain some 2-coset. However, it cannot contain any 2-coset, since knowing the image of any two points in a permutation π is not enough to certify that $f(\pi) = 1$. (We leave it to the reader to perform the necessary case analysis.)

This counterexample can be generalized to any even $k \geq 2$ and $n \geq 2(k + 1)$:

$$\begin{aligned} \llbracket \pi(\{1, \dots, k + 1\}) = \{1, \dots, k + 1\} \text{ or } \pi(\{1, \dots, k + 1\}) \cap \{1, \dots, k + 1\} = \emptyset \rrbracket = \\ \sum_{S \subsetneq \{1, \dots, k + 1\}} (-1)^{|S|} \llbracket \pi(S) \subset \{1, \dots, k + 1\} \rrbracket. \end{aligned}$$

This is a function in V_k whose support contains no k -coset.

We describe other counterexamples in Section 5.

4 Counterexample to Theorem 29

The proof of Theorem 27 relies on a claimed generalization of Birkhoff's theorem attributed to Benabbas, Friedgut, and Pilpel. Birkhoff's theorem states that every bistochastic matrix is a convex combination of permutation matrices. Theorem 29 of [EFP11] purports to generalize Birkhoff's theorem to k dimensions. Since our counterexample is two-dimensional, we will concentrate on the case $k = 2$.

A 2-bistochastic matrix M is an $n(n-1) \times n(n-1)$ matrix whose rows and columns are indexed by pairs of indices, such that the following two properties hold:

- (S1) There exist an $n \times n$ bistochastic matrix $R = (r_{i,j})$ and n^2 bistochastic matrices $M_{i,j}$ of dimension $(n-1) \times (n-1)$ whose rows and vertices are indexed by $[n] \setminus \{i\}, [n] \setminus \{j\}$ (respectively) such that $M((i, i'), (j, j')) = r_{i,j} M_{i,j}(i', j')$.
- (S2) There exist an $n \times n$ bistochastic matrix $R' = (r_{i',j'})$ and n^2 bistochastic matrices $M'_{i',j'}$ of dimension $(n-1) \times (n-1)$ whose rows and vertices are indexed by $[n] \setminus \{i'\}, [n] \setminus \{j'\}$ (respectively) such that $M((i, i'), (j, j')) = r_{i',j'} M'_{i',j'}(i, j)$.

Every permutation $\pi \in S_n$ gives rise to a 2-bistochastic matrix M_π given by $M_\pi((i, i'), (j, j')) = \mathbb{1}[\pi(i) = j \text{ and } \pi(i') = j']$. We can now state Theorem 29 in the special case $k = 2$.

Theorem 29 *An $n(n-1) \times n(n-1)$ matrix M is 2-bistochastic if and only if it is a convex combination of $n(n-1) \times n(n-1)$ matrices induced by permutations of $[n]$.*

The following matrix is a counterexample to this theorem, for $n = 4$ (the theorem holds for smaller n):

	12	13	14	21	23	24	31	32	34	41	42	43
12	0	1/4	0	0	0	0	0	0	1/4	1/4	1/4	0
13	0	0	1/4	0	0	0	1/4	0	0	0	1/4	1/4
14	1/4	0	0	0	0	0	0	1/4	0	1/4	0	1/4
21	0	0	1/4	1/4	0	0	0	0	1/4	0	0	1/4
23	0	1/4	0	0	0	1/4	0	1/4	0	1/4	0	0
24	1/4	0	0	0	1/4	0	1/4	0	0	0	1/4	0
31	0	0	1/4	0	1/4	0	0	0	1/4	1/4	0	0
32	1/4	0	0	0	0	1/4	1/4	0	0	0	0	1/4
34	0	1/4	0	1/4	0	0	0	1/4	0	0	1/4	0
41	0	1/4	0	0	0	1/2	1/4	0	0	0	0	0
41	0	0	1/4	1/4	1/4	0	0	1/4	0	0	0	0
43	1/4	0	0	1/4	1/4	0	0	0	1/4	0	0	0
	21	31	41	12	32	42	13	23	43	14	24	34
21	1/4	0	0	0	0	0	0	0	1/4	1/4	0	1/4
31	0	0	1/4	0	0	0	0	1/4	0	1/4	0	1/4
41	0	1/4	0	0	0	0	1/4	0	0	0	1/2	0
12	0	0	1/4	0	0	1/4	1/4	0	0	0	0	1/4
32	0	1/4	0	1/4	0	0	0	0	1/4	0	1/4	0
42	1/4	0	0	0	1/4	0	0	1/4	0	1/4	0	0
13	0	1/4	0	0	0	1/4	0	0	1/4	1/4	0	0
23	0	0	1/4	0	1/4	0	1/4	0	0	0	1/4	0
43	1/4	0	0	1/4	0	0	0	1/4	0	0	0	1/4
14	0	0	1/4	1/4	1/4	0	0	0	1/4	0	0	0
24	0	1/4	0	1/4	0	1/4	0	1/4	0	0	0	0
34	1/4	0	0	0	1/4	1/4	1/4	0	0	0	0	0

The reader can check that this matrix (given in two different orderings of the rows and columns) is 2-bistochastic. According to Theorem 29, it should be a convex combination of matrices induced by permutations. However, its support does not contain the support of any matrix induced by a permutation.

We go on to pinpoint the mistake in the proof of Theorem 29; what follows uses the same notation as the original proof. Let M be a 2-bistochastic matrix. According to condition (S1), there are bistochastic matrices $R = (r_{i,j})$ and $M_{i,j}$ such that $M((i, i'), (j, j')) = r_{i,j}M_{i,j}(i', j')$. Since R is bistochastic, according to Birkhoff's theorem it is a convex combination of permutation matrices. If P is one of these permutation matrices, then we can write $R = sP + (1-s)T$ for some $s \in (0, 1]$. The proof then considers the modified matrix \tilde{M} (*my notation*) defined by $\tilde{M}((i, i'), (j, j')) = P(i, j)M_{i,j}(i', j')$. Under the (tacit) assumption that \tilde{M} is 2-bistochastic, the proof goes on to show that \tilde{M} is the matrix induced by the permutation corresponding to P . However, while \tilde{M} certainly satisfies (S1), it need not satisfy (S2). For a concrete example, consider the matrix M above. The corresponding matrix R is

$$\begin{pmatrix} 1/4 & 0 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Denote the first permutation matrix by P . If we replace R by P then we get the following matrix:

	12	13	14	21	23	24	31	32	34	41	42	43
12	0	0	0	0	0	0	0	0	0	1/2	1/2	0
13	0	0	0	0	0	0	0	0	0	0	1/2	1/2
14	0	0	0	0	0	0	0	0	0	1/2	0	1/2
21	0	0	0	0	0	0	0	0	1	0	0	0
23	0	0	0	0	0	0	0	1	0	0	0	0
24	0	0	0	0	0	0	1	0	0	0	0	0
31	0	0	0	0	0	0	0	1	0	0	0	0
32	0	0	0	0	0	0	0	0	1	0	0	0
34	0	0	0	1	0	0	0	0	0	0	0	0
41	0	1	0	0	0	0	0	0	0	0	0	0
42	0	0	1	0	0	0	0	0	0	0	0	0
43	1	0	0	0	0	0	0	0	0	0	0	0

	21	31	41	12	32	42	13	23	43	14	24	34
21	0	0	0	0	0	0	0	0	0	0	0	1
31	0	0	0	0	0	0	0	1	0	0	0	0
41	0	0	0	0	0	0	1	0	0	0	0	0
12	0	0	1/2	0	0	1/2	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	1	0
42	0	0	0	0	0	0	0	0	0	1	0	0
13	0	0	0	0	0	1/2	0	0	1/2	0	0	0
23	0	0	0	0	1	0	0	0	0	0	0	0
43	0	0	0	1	0	0	0	0	0	0	0	0
14	0	0	1/2	0	0	0	0	0	1/2	0	0	0
24	0	1	0	0	0	0	0	0	0	0	0	0
34	1	0	0	0	0	0	0	0	0	0	0	0

As can be seen, this matrix satisfies (S1) but not (S2).

5 Final remarks

The counterexample described to Theorem 27 arose while trying to prove a similar theorem on the *slice* $J(n, k) = \{x \in \{0, 1\}^n : \sum_i x_i = k\}$. Specifically, we were interested in understanding the structure of Boolean functions of degree d (the degree of a function on $J(n, k)$ is the minimal degree of a polynomial defining it). Every Boolean function f on the slice $J(n, k)$ can be lifted to a Boolean function F on S_n using the formula

$$F(\pi) = f(\pi(\{1, \dots, k\})).$$

Moreover, if f has degree d then $F \in V_d$. It is then a simple exercise to deduce the following theorem from Theorem 27:

Theorem 27J *If f is a nonnegative function on $J(n, k)$ of degree d then f can be written as a nonnegative combination of degree- d monomials in the variables $x_1, \dots, x_n, 1 - x_1, \dots, 1 - x_n$. Moreover, if f is a Boolean function on $J(n, k)$ of degree d then f can be written as a sum of degree- d monomials (in the same variables), which are necessarily “disjoint” in the sense that no two can evaluate to 1 at the same time.*

This theorem fails for the following function, which is where our counterexample to Theorem 27 comes from:

$$f = \llbracket x_1 = x_2 = x_3 \rrbracket = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3.$$

Curiously, it does hold for the negation of f :

$$1 - f = x_1(1 - x_2) + x_2(1 - x_3) + x_3(1 - x_1).$$

As noticed by David Ellis, this example can be extended to any even degree.

It is natural to ask whether Theorem 27 or Theorem 27J can be corrected. Here are two possible ways to correct Theorem 27J, in the special case of Boolean functions:

1. For every d there is a constant s_d such that every Boolean function on $J(n, k)$ of degree d can be written as a sum monomials of degree s_d .
2. For every d there is a constant r_d such that for every Boolean function f on $J(n, k)$ of degree d , either f or $1 - f$ can be written as a sum of monomials of degree r_d .

For a Boolean function f on $\{0, 1\}^n$, the minimal s such that f can be written as a sum of monomials of degree s is known as its one-sided *unambiguous certificate complexity*, which is known to be polynomially related to the degree [Gö15, BD17]. The analog of the parameter r is known as the two-sided unambiguous certificate complexity, and is also polynomial related to the degree [Gö15, BD17]. Therefore for the Boolean cube we know that $s_d, r_d = d^{\Theta(1)}$.

A classical example showing that $r_2 > 2$ (on both the Boolean cube and $J(n, k)$ for appropriate n, k) is the 4-sortedness function

$$f(x_1, x_2, x_3, x_4) = \llbracket x_1 \leq x_2 \leq x_3 \leq x_4 \text{ or } x_1 \geq x_2 \geq x_3 \geq x_4 \rrbracket.$$

While not apparent from this definition, f has degree 2. It is not hard to check that the unambiguous certificate complexity of both f and $1 - f$ is larger than 2. By iterating this function, we get the lower bound $r_d = d^{\Omega(1)}$.

Finally, let us mention, that an approximate version of Theorem 27 does hold for sparse Boolean functions, namely the main result of [EFF15a]. Roughly speaking, this result states that if f is a Boolean function close to V_d (in L^2) and $\mathbb{E}[f] = O(1/n^d)$ then f is close to a union of d -cosets. For similar results in the case $d = 1$, see [EFF15b, EFF15c].

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