The Complexity of the Comparator Circuit Value Problem

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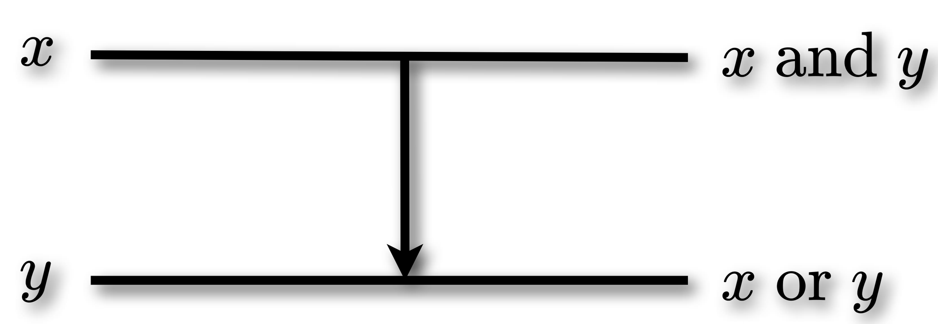
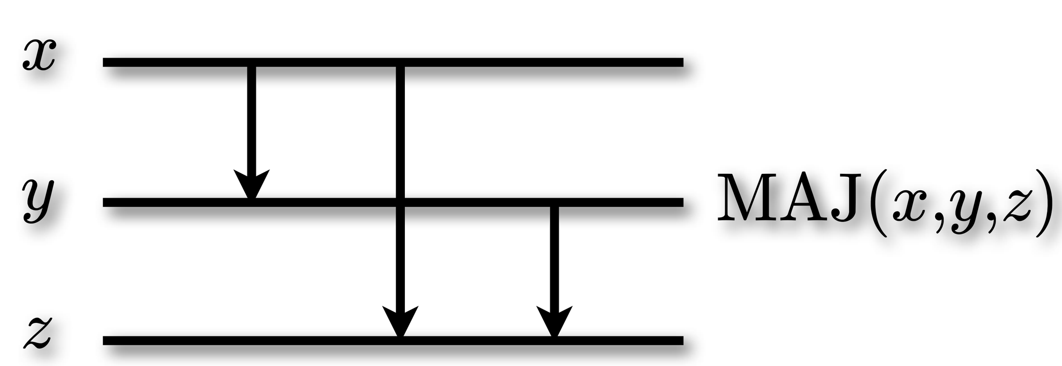
The Complexity Class CC

Comparator circuits are composed of comparator gates:



Comparator Gate

Wires are initialized with constants, inputs or negated inputs.
Output is read from a designated output wire.
Wires cannot split!



Comparator Circuit

Comparator Circuit Value Problem (CCVP):

Given a comparator circuit and an assignment to the inputs, determine the value of the output. [Mayr & Subramanian '92]

The complexity class CC

Definitions as closure of CCVP:

1. All languages AC^0-Turing reducible to CCVP.
2. All languages AC^0-many-one reducible to CCVP.
3. All languages NL-many-one reducible to CCVP. [Mayr & Sub. '92]

Definitions as uniform comparator circuits:

4. Predicates computed by uniform polynomial-size comparator circuits with input wires initialized by constants, inputs and negated inputs.
5. Predicates computed by uniform polynomial-size comparator circuits with negation gates and input wires initialized by constants and inputs.

Containment Relations and Complete Problems

$$\boxed{\mathrm{NL} \subseteq \mathrm{CC} \subseteq \mathrm{P}}$$

[Mayr & Sub. '92]

Solving directed reachability using comparator circuits:
(our simplified proof)

- Repeat n times:
  - Pebble source node
  - For each edge $x \to y$:
    - Move pebble from x to y if possible (★)
- Check if target is pebbled

Comparator circuit depends on input graph, but can be constructed from input graph in AC^0.

Operation ★ is implemented by a comparator.

Complete problems:

- Comparator circuit value problem
- Lexicographically-first maximal matching: [Mayr & Subramanian '92]

Given a bipartite graph (V,W,E), greedily match vertices in V to first unmatched vertex in W.
Is a given vertex matched?
Is a given edge part of the matching?

- Stable marriage problem: [Subramanian '94]

Input: n men and n women, each with a preference order on people of the opposite sex.
A stable marriage is a perfect matching in which no two unmatched people prefer each other over their current partners.
Are two people matched in the woman-optimal solution?

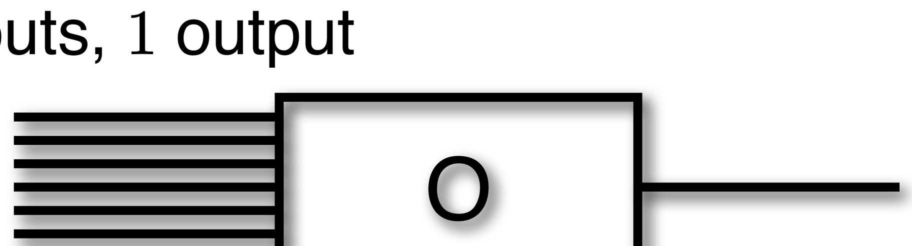
Gale & Shapley: stable marriage always exists, algorithm.
Subramanian: fixed-point algorithm using comparator circuits.

Oracle Separations

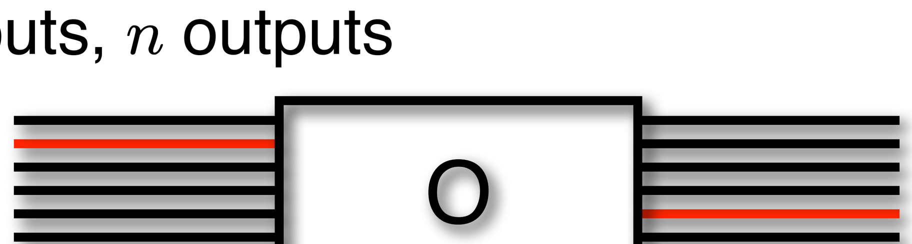
Conjecture: CC and NC are incomparable. Implies NL≠P.
Evidence: oracle separations.

Oracle models: (separations work for both)

(1) n inputs, 1 output



(2) n inputs, n outputs



Satisfies 1-flip property:
If one input bit is flipped, one output bit is flipped.

A function in $CC^O \setminus NC^O$:

Oracle encodes a function f: $\{0,1\}^n \to \{0,1\}^n$.
Problem: compute bit 0 of $f^{(\sqrt{n})}(0)$.
Requires depth $\sqrt{n}$, which is easy for CC^O.

A function in $NC^O \setminus CC^O$:

Oracle encodes a sequence of functions
$f_1,\ldots,f_m$: $\{0,1\}^{4n} \to \{0,1\}^n$, where $m = \log^2 n$.
Each function expects an input of the form

$$x_1x_1x_1x_1x_2x_2x_2x_2\ldots x_nx_nx_nx_n$$

Problem: compute parity of $f_m \circ \cdots \circ f_2 \circ f_1$.
Requires fan-out, which is easy for NC^O.
Lower bound for CC^O uses 1-flip property.