Uniqueness for 2-Intersecting Families of Permutations and Perfect Matchings

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Abstract
We give a characterization of the largest 2-intersecting families of permutations of \(\{1, 2, \ldots, n\}\) and perfect matchings of the complete graph \(K_{2n}\) for all \(n \geq 2\).

1 Introduction

Erdős–Ko–Rado theory \cite{EKR61} studies intersecting families of objects. One of the high points of the theory is the Ahlswede–Khachatrian theorem \cite{AK97, AK99}.

\textbf{Theorem 1.1.} Let \(F\) be a subset of \(\binom{[n]}{k}\) (the collection of all subsets of \([n] := \{1, \ldots, n\}\) of size \(k\)) which is \(t\)-intersecting: every \(A, B \in F\) satisfy \(|A \cap B| \geq t\). Then

\[
|F| \leq \max_{r \leq (n-t)/2} \left\{|A \subseteq \binom{[n]}{k} : |A \cap [t+2r]| \geq t+r\right\}.
\]

Furthermore, if \(F\) achieves this bound, then

\[
F = \left\{A \subseteq \binom{[n]}{k} : |A \cap S| \geq t+r\right\}
\]

for some \(r \leq (n-t)/2\) and some \(S \subseteq [n]\) of size \(t+2r\).

The theorem consists of two statements: an \textit{upper bound} on the size of \(t\)-intersecting families, and a characterization of the extremal families. The latter part is known as \textit{uniqueness}.

The Ahlswede–Khachatrian theorem is about intersecting families of sets. In this paper, we will be interested in intersecting families of permutations and perfect matchings.

\textbf{Permutations} Let \(S_n\) be the group of all permutations of \([n]\). Two permutations \(\alpha, \beta \in S_n\) are \(t\)-\textit{intersecting} if there exist \(t\) distinct indices \(i_1, \ldots, i_t \in [n]\) such that \(\alpha(i_1) = \beta(i_1), \ldots, \alpha(i_t) = \beta(i_t)\). Ellis, Friedgut and Pilpel \cite{EFP11} conjectured that the Ahlswede–Khachatrian theorem extends to intersecting families of permutations.
**Conjecture 1.2.** Let \( F \) be a subset of \( S_n \) which is \( t \)-intersecting. Then
\[
|F| \leq \max_{r \leq (n-t)/2} \left| \{ \alpha \in S_n : \alpha(i) = i \text{ for at least } t + r \text{ many } i \in [t+2r] \} \right|.
\]
Furthermore, if \( F \) achieves this bound, then
\[
F = \{ \alpha \in S_n : \alpha(i_s) = j_s \text{ for at least } t + r \text{ many } s \in [t+2r] \}
\]
for some \( r \leq (n-t)/2 \) and some distinct \( i_1, \ldots, i_{t+2r}, j_1, \ldots, j_{t+2r} \in [n] \).

When \( t \leq 3 \), the maximum is always attained at \( r = 0 \), in which case the upper bound is \((n-t)!\), and the conjectured extremal families are the \( t \)-cosets.

Lindsey [LD77] proved the upper bound part of Conjecture 1.2 in the special case \( t = 1 \) by noticing that the \( n \) cyclic rotations of any fixed permutation are pairwise non-1-intersecting. Proving uniqueness in this case proved a lot harder, but eventually many proofs were found [CK03, LM04, GM09, EFP11].

The case \( t > 1 \) is significantly harder. For every \( t > 1 \), Ellis, Friedgut and Pilpel [EFP11] proved that the upper bound in Conjecture 1.2 holds for large enough \( n \), and Ellis [Ell12a] proved that the uniqueness part of Conjecture 1.2 holds for large enough \( n \). Ellis, Friedgut and Pilpel used a spectral technique based on the so-called Hoffman bound (closely related to the Lovász theta function and to the linear programming bound in coding theory), which has found many other applications in Erdős–Ko–Rado theory [GM10].

Recently, Meagher and Razafimahatratra [MR21], refining the techniques of Ellis, Friedgut and Pilpel, proved that 2-intersecting subsets of \( S_n \) contain at most \((n-2)!\) permutations, thus verifying the upper bound part of Conjecture 1.2 for \( t = 2 \) and all \( n \). However, they were unable to characterize the extremal families in this case. In this paper, we show that all such families are 2-cosets, thus verifying the uniqueness part of Conjecture 1.2 for \( t = 2 \) and all \( n \).

**Theorem 1.3.** Let \( n \geq 2 \). If \( F \) is a 2-intersecting subset of \( S_n \) of size \((n-2)!\), then there exist \( i_1 \neq i_2 \) and \( j_1 \neq j_2 \) such that
\[
F = \{ \alpha \in S_n : \alpha(i_1) = j_1 \text{ and } \alpha(i_2) = j_2 \}.
\]

**Perfect matchings** So far we have considered intersecting families of permutations. A related area of study is intersecting families of perfect matchings of the complete graph \( K_{2n} \). This can be seen as the non-bipartite analog of the symmetric group, which is the set of perfect matchings in the complete bipartite graph \( K_{n,n} \).

A simple argument shows that a 1-intersecting family of matchings contains at most \((2n-3)!! = (2n-3)(2n-5) \cdots (1) \) perfect matchings. More difficult arguments [GM17, Lin17, MM05, DFL+21] show that this is attained uniquely by 1-cosets.

Lindsey [LD18a, LD18b] extended the arguments of Ellis, Friedgut and Pilpel [EFP11, Ell12a] to the setting of perfect matchings, proving that for all \( t \), the maximum size \( t \)-intersecting families are \( t \)-cosets, for large enough \( n \).

Recently, Fallat, Meagher and Shirazi showed that the maximum size of a 2-intersecting family is at most \((2n-5)!!\) for all \( n \). Extending our arguments in the setting of the symmetric group, we prove an analog of Theorem 1.3 in the setting of the perfect matching scheme.

**Theorem 1.4.** Let \( n \geq 2 \). If \( F \) is a 2-intersecting subset of \( \mathcal{M}_{2n} \) of size \((2n-5)!!\), then there exist distinct \( i_1, j_1, i_2, j_2 \) such that \( F \) consists of all perfect matchings containing the edges \( \{i_1,j_1\}, \{i_2,j_2\} \).

**On the proof** The spectral technique used by Ellis, Friedgut and Pilpel shows that for every \( t \geq 1 \), if \( n \) is large enough and \( F \subseteq S_n \) is a \( t \)-intersecting family of size \((n-t)!\), then the characteristic function of \( F \), denoted as \( 1_F \), has degree at most \( t \). This means that \( 1_F \) can be expressed as a polynomial of degree at most \( t \) in the Boolean variables \( x_{ij} \), where \( x_{ij} = 1 \) if the input permutation maps \( i \) to \( j \). Equivalently, \( x_{ij} \) is the \((i,j)\)'th entry of the permutation matrix representing the input permutation.

Similarly, Meagher and Razafimahatratra show that if \( n \geq 5 \) and \( F \subseteq S_n \) is a 2-intersecting family of size \((n-2)!\), then the characteristic function of \( F \) has degree at most 2. This is the starting point of this work.
Ellis, Friedgut and Pilpel used polyhedral techniques (chiefly, the Birkhoff–von Neumann theorem on bistochastic matrices) to show that Boolean degree 1 functions are dictators, that is, they either depend only on \(\alpha(i)\) for some \(i \in [n]\) (where \(\alpha\) is the input permutation), or they depend only on \(\alpha^{-1}(j)\) for some \(j \in [n]\). The only 1-intersecting dictators are 1-cosets, and this provides a proof of the uniqueness part of Conjecture 1.2 when \(t = 1\).

Similar techniques do not work for larger degree \[\text{FIl17}\]. Instead, we turn to the theory of complexity measures of Boolean functions \[\text{BDW02}\], which was recently generalized to the symmetric group and to the perfect matching scheme by Dafni et al. \[\text{DFL+21}\]. Using these techniques, Dafni et al. give a simple proof of the uniqueness part of Conjecture 1.2 as well as of its analog for the perfect matching scheme, for large \(n\).

The chief tool used by Dafni et al., which we also employ in our proof, is certificate complexity. Let \(f : S_n \to \{0,1\}\) be a Boolean function, and let \(\alpha \in S_n\) be such that \(f(\alpha) = b\). A certificate for \(\alpha\) is a subset \(\{i_1, \ldots, i_m\} \subseteq [n]\) such that \(f(\beta) = b\) whenever \(\beta(i_1) = \alpha(i_1), \ldots, \beta(i_m) = \alpha(i_m)\). The idea is that in order to verify that \(f(\alpha) = b\), it suffices to check the value of \(\alpha(i_1), \ldots, \alpha(i_m)\). The certificate complexity of \(\alpha\) is the minimum size of a certificate for \(\alpha\), and the certificate complexity of \(f\) is the maximum, over all \(\alpha \in S_n\), of the certificate complexity of \(\alpha\). We denote the certificate complexity of \(f\) by \(C(f)\).

Suppose that \(f\) is the characteristic function of a 2-intersecting subset of \(S_n\). When \(C(f) \leq n - 2\), we give a structure theorem for \(f\) which suffices to bound the size of the family away from \((n - 2)!\) unless \(C(f) = 2\), in which case \(f\) is the characteristic function of a 2-coset. A simple argument (using another complexity measure, sensitivity) shows that if \(n \geq 8\) and \(f : S_n \to \{0,1\}\) has degree at most 2 then \(C(f) \leq n - 2\), allowing us to apply the preceding argument. Finally, we handle the case \(n \leq 7\) using exhaustive search, employing an algorithm for enumerating maximum cliques.

Similar techniques work in the case of the perfect matching scheme, with one complication: the proof of our structure theorem relies on the fact that Boolean degree 1 functions on the symmetric group have certificate complexity 1, but this fails for the perfect matching scheme. Using the maximality of maximum-size 2-intersecting families, we are able to overcome this hurdle.

**Future research** Conjecture 1.2 is about families of permutations in which any two permutations agree on the images of \(t\) points. A related question concerns \(t\)-setwise-intersecting families of permutations, in which any two permutations \(\alpha, \beta\) agree on the image of a set of size \(t\): \(\{\alpha(i_1), \ldots, \alpha(i_t)\} = \{\beta(i_1), \ldots, \beta(i_t)\}\) for a set of size \(t\).

Ellis \[\text{Ell12}\] showed that for every \(t\), if \(n\) is large enough then the maximum size of a \(t\)-setwise-intersecting subset of \(S_n\) is \(t!(n-t)!\), and this is achieved uniquely by families of the form

\[
\{\alpha \in S_n : \{\alpha(i_1), \ldots, \alpha(i_t)\} = \{j_1, \ldots, j_t\}\},
\]

which we call \(t\)-setwise-cosets.

Meagher and Razafimahatrata \[\text{MR21}\] showed that when \(t = 2\), the upper bound holds for all \(n \geq 2\), and moreover, if \(n \geq 4\) and \(f\) is the characteristic function of a 2-setwise-intersecting family of size \(2(n-2)!\), then \(f\) has degree at most 2 and additionally satisfies \(f = f^{(n-2,1,1)} = 0\) (see Section 2.1.1 for an explanation of this notation). In other words, \(f = f^{\lambda} \neq 0\) only for \(\lambda = (n), (n - 1, 1), (n - 2, 2)\).

Behajaina, Maleki, Rasoamanana and Razafimahatrata \[\text{BMRR21}\] showed that when \(t = 3\), the upper bound holds for all \(n \geq 11\), and moreover, if \(f\) is the characteristic function of a 3-setwise-intersecting family of size \(6(n-3)!\), then \(f = f^{\lambda} \neq 0\) only for \(\lambda = (n), (n - 1, 1), (n - 2, 2), (n - 3, 3)\).

In both cases, the authors were unable to classify the extremal families. Can we extend our techniques to show that the extremal families are \(t\)-setwise-cosets? The first step in this direction was already taken by the second author in her master’s thesis \[\text{DaP22}\], who showed that this holds for large \(n\). One of the difficulties in obtaining bounds for small \(n\) is taking advantage of the characterization of extremal \(f\), which is stronger than a mere degree bound.

We mention in passing two more challenges. One is to extend the theory of setwise-intersecting families from permutations to perfect matchings. A \(t\)-setwise intersecting family of perfect matchings is one in which for any two perfect matchings \(m_1, m_2\) there are two sets \(A, B\) of size \(t\) such that both \(m_1, m_2\) match the
vertices in \( A \) to vertices in \( B \). Another possible generalization is to require the existence of a single set \( C \) of size \( 2t \) such that both \( m_1 \) and \( m_2 \) match vertices in \( C \) to vertices in \( C \).

The other is a common generalization of \( t \)-intersecting families and \( t \)-setwise-intersecting families. Given a partition \( \lambda = \lambda_1, \ldots, \lambda_m \), a \( \lambda \)-intersecting family of permutations is one in which for any two permutations \( \alpha, \beta \) there are disjoint sets \( A_1, \ldots, A_m \) of sizes \( \lambda_1, \ldots, \lambda_m \) such that \( \alpha(A_1) = \beta(A_1), \ldots, \alpha(A_m) = \beta(A_m) \). A \( t \)-intersecting family corresponds to \( \lambda = (1^t) \), and a \( t \)-setwise-intersecting family corresponds to \( \lambda = (t) \).

**Structure of the paper**  We start with a few preliminaries in Section 1. Apart from introducing several key definitions and results, we also prove several results appearing in a more general form in [DFL+21], in order to keep the paper self-contained. Theorem 1.3 is then proved in Section 3, and Theorem 1.4 in Section 4.

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## 2 Preliminaries

### 2.1 Symmetric group

For integer \( n \geq 1 \), the **symmetric group** \( S_n \) is the collection of all permutations of \( [n] = \{1, \ldots, n\} \). We denote the identity permutation by \( \text{id} \).

A \( t \)-coset of \( S_n \) is a set of the form

\[
\{ \alpha \in S_n : \alpha(i_1) = j_1, \ldots, \alpha(i_t) = j_t \},
\]

where \( i_1, \ldots, i_t \in [n] \) are distinct and \( j_1, \ldots, j_t \in [n] \) are distinct. A \( t \)-coset contains \((n - t)!\) permutations.

We use the term **coset** for a 1-coset. For \( i, j \in [n] \), the coset \([i \mapsto j]\) consists of all permutations sending \( i \) to \( j \):

\[
[i \mapsto j] = \{ \alpha \in S_n : \alpha(i) = j \}.
\]

Two permutations \( \alpha, \beta \in S_n \) are **t-intersecting** if there is a \( t \)-coset which contains both of them. Equivalently, the two permutations \( t \)-intersect if there are distinct indices \( i_1, \ldots, i_t \in [n] \) such that \( \alpha(i_1) = \beta(i_1), \ldots, \alpha(i_t) = \beta(i_t) \).

A subset \( \mathcal{F} \subseteq S_n \) is **t-intersecting** if every pair of permutations in \( \mathcal{F} \) are \( t \)-intersecting.

#### 2.1.1 Degree

We can represent permutations \( \alpha \in S_n \) using \( n^2 \) variables \( x_{ij} \) whose semantics are: \( x_{ij} = 1 \) if \( \alpha(i) = j \), and \( x_{ij} = 0 \) otherwise. The **degree** of a function \( f : S_n \to \mathbb{R} \), denoted \( \deg f \), is the minimal \( d \) such that \( f \) can be represented as a degree \( d \) polynomial over the variables \( x_{ij} \). For example, the characteristic function of a \( t \)-coset has degree at most \( t \), since it can be represented by the polynomial \( x_{i_1 j_1} \cdots x_{i_t j_t} \).

An equivalent way to define degree is via the representation theory of the symmetric group. Let \( \mathbb{R}[S_n] \) be the vector space of all real-valued functions over the symmetric group. Representation theory gives an orthogonal decomposition (with respect to the inner product \( \langle f, g \rangle = \sum_{\alpha \in S_n} f(\alpha)g(\alpha) \))

\[
\mathbb{R}[S_n] = \bigoplus_{\lambda \vdash n} V^\lambda,
\]

where \( \lambda \) goes over all integer partitions of \( n \), and \( V^\lambda \) are certain subspaces known as **isotypic components**, which we define explicitly in Section 2.1.3. Accordingly, every function \( f : S_n \to \mathbb{R} \) has a unique decomposition

\[
f = \sum_{\lambda \vdash n} f^\lambda,
\]

4
where \( f = \lambda \in V^\lambda \). Ellis, Friedgut and Pilpel [EFPII] Theorem 7 showed that \( d = \max d \) such that \( f = \lambda \neq 0 \) for some \( \lambda \) satisfying \( \lambda_1 = n - d \). In particular, every function has degree at most \( n - 1 \).

We can now formally state the main result of Meagher and Razafimahatratra [MR21] Corollary 5.5.

**Theorem 2.1.** If \( n \geq 5 \) then any 2-intersecting family \( F \subseteq S_n \) contains at most \( (n - 2)! \) permutations. Furthermore, if \( f : S_n \to \{0, 1\} \) is the characteristic vector of a 2-intersecting family of size \( (n - 2)! \), then \( \deg f \leq 2 \).

We will also need a result of Ellis, Friedgut and Pilpel [EFPII] classifying degree 1 functions.

**Theorem 2.2** ([EFPIII Corollary 2]). If \( f : S_n \to \{0, 1\} \) has degree at most 1 then either

\[
\sum_{j \in J} x_{ij}
\]

for some \( i \in [n] \) and \( J \subseteq [n] \), or

\[
\sum_{i \in I} x_{ij}
\]

for some \( I \subseteq [n] \) and \( j \in [n] \).

See also [DFL+21] Theorem 6.1 for an alternative (but very similar) proof.

### 2.1.2 Certificate complexity

Fix \( n \). A **certificate** is a set \( C = \{(i_1, j_1), \ldots, (i_m, j_m)\} \subseteq [n] \times [n] \). A permutation \( \alpha \in S_n \) satisfies the certificate \( C \) if \( \alpha(i_1) = j_1, \ldots, \alpha(i_m) = j_m \). The set of permutations satisfying a certificate \( C \) is either empty or a \(|C|\)-coset.

We sometimes think of a permutation \( \alpha \in S_n \) as the certificate \( \{(1, \alpha(1)), \ldots, (n, \alpha(n))\} \), which we call the **certificate representation** of \( \alpha \).

A **Boolean function** is a function \( f : S_n \to \{0, 1\} \). Given \( \alpha \in S_n \) such that \( f(\alpha) = b \), a **certificate for** \( \alpha \) (with respect to \( f \)) is a certificate \( C \) such that:

(i) \( \alpha \) satisfies \( C \).

(ii) If \( \beta \) satisfies \( C \) then \( f(\beta) = b \).

Intuitively, in order to certify that \( f(\alpha) = b \), it suffices to verify that \( \alpha \) satisfies \( C \).

The **certificate complexity** of \( \alpha \), denoted \( C(f, \alpha) \), is the minimum size of a certificate for \( \alpha \). A certificate for \( \alpha \) of this size is known as a **minimum certificate**. Since every permutation is determined by its values on the points \( 1, \ldots, n - 1 \), the certificate complexity of \( \alpha \) is always at most \( n - 1 \).

The **certificate complexity** of \( f \), denoted \( C(f) \), is \( \max_{\alpha \in S_n} C(f, \alpha) \). The certificate complexity of \( f \) is always at most \( n - 1 \).

Dafni et al. [DFL+21] Theorem 3.1 showed that certificate complexity is polynomially related to the degree: \( C(f) = O((\deg f)^9) \) and \( \deg f = O(C(f)^4) \).

In the special case of degree 1, Theorem 2.2 implies that \( C(f) \leq 1 \).

**Lemma 2.3.** If \( f : S_n \to \{0, 1\} \) has degree at most 1 then \( C(f) \leq 1 \).

**Proof.** According to Theorem 2.2, either \( f = \sum_{j \in J} x_{ij} \) for some \( i \in [n] \) and \( J \subseteq [n] \), or \( f = \sum_{i \in I} x_{ij} \) for some \( I \subseteq [n] \) and \( j \in [n] \). In the former case, every \( \alpha \in S_n \) has the certificate \( \{(i, \alpha(i))\} \), and in the latter case, every \( \alpha \in S_n \) has the certificate \( \{((\alpha^{-1}(j)), j)\} \).

The following lemma, which essentially follows from [DFL+21] Lemma 7.3, shows that if \( F \) is a 2-intersecting family then minimum certificates of any two permutations in \( F \) must 2-intersect, unless \( C(f) = n - 1 \).
Lemma 2.4. Let $f : S_n \to \{0, 1\}$ be the characteristic function of a 2-intersecting family, and suppose that $C(f) \leq n - 2$. If $f(\alpha) = f(\beta) = 1$ and $C_\alpha, C_\beta$ are minimum certificates for $\alpha, \beta$ (respectively), then $|C_\alpha \cap C_\beta| \geq 2$.

Proof. We will show that there exist $\alpha' \in S_n$ satisfying $C_\alpha$ and $\beta' \in S_n$ satisfying $C_\beta$ such that $\alpha' \cap \beta' = C_\alpha \cap C_\beta$, where we identify $\alpha', \beta'$ with their certificate representations. Since $f(\alpha') = f(\beta') = 1$ and $f$ is the characteristic function of a 2-intersecting family, we have $|\alpha' \cap \beta'| \geq 2$, and so $|C_\alpha \cap C_\beta| \geq 2$.

We start by constructing $\alpha'$ satisfying $C_\alpha$ such that $\alpha' \cap C_\beta = C_\alpha \cap C_\beta$. Let $C_\alpha = \{(i_1, j_1), \ldots, (i_m, j_m)\}$, where $m \leq n - 2$, let $i'_1, \ldots, i'_{n-m}$ be the $i$-indices not mentioned in $C_\alpha$, and let $j'_1, \ldots, j'_{n-m}$ be the $j$-indices not mentioned in $C_\alpha$.

For each $i'_i$, there is at most one $j'_j$ such that $(i'_i, j'_j) \in C_\beta$. We can therefore arrange the indices in such a way that if $(i'_i, j'_j) \subseteq C_\beta$ then $s = t$. We can now define $\alpha'$ explicitly:

$$\alpha'(i_1) = j_1, \ldots, \alpha'(i_m) = j_m, \alpha'(i'_1) = j'_2, \ldots, \alpha'(i'_{n-m-1}) = j'_{n-m}, \alpha'(i'_{n-m}) = j'_1.$$ 

The same argument (replacing $C_\alpha, C_\beta$ by $C_\beta, C_\alpha$) shows that we can find $\beta'$ satisfying $C_\beta$ such that $\alpha' \cap \beta' = C_\alpha \cap C_\beta = C_\alpha \cap C_\beta$, completing the proof.

When $C(f) = n - 1$, the conclusion of Lemma 2.4 indeed fails. For example, consider the family $F \subseteq S_5$ given by $F = \{id, (1 2 3)\}$, and the minimum certificates $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $\{(1, 2), (2, 3), (3, 1), (4, 4)\}$.

Using the concept of sensitivity, we can rule out the case $C(f) = n - 1$ for functions of degree at most 2. The proof is based on the technique used to prove [DFL+21] Lemma 3.6.

Lemma 2.5. If $n \geq 8$ and $f : S_n \to \{0, 1\}$ has degree at most 2, then $C(f) \leq n - 2$.

Proof. The proof is by contradiction. Suppose that $n \geq 8$, that $f : S_n \to \{0, 1\}$ has degree at most 2, and that $C(f) = n - 1$. Without loss of generality, $C(f, id) = n - 1$ and $f(id) = 0$. Then $f((i j)) = 1$ for all $i \neq j \in [n]$ (here $(i j)$ is the transposition switching $i$ and $j$), since otherwise $\{(k, k) : k \neq i, j\}$ would be a certificate for id of size $n - 2$.

We construct a function $g : \{0, 1\}^4 \to \{0, 1\}$ as follows:

$$g(y_1, y_2, y_3, y_4) = f((1 2)^{y_1}(3 4)^{y_2}(5 6)^{y_3}(7 8)^{y_4}).$$

Here $(i j)^0 = id$, $(i j)^1 = (i j)$, and the input to $f$ is the product of four powers of this kind.

Since $f$ has degree at most 2, it can be written as a polynomial of degree at most 2 in the variables $x_{ij}$. Given $y_1, y_2, y_3, y_4$, the value of the variables $x_{ij}$ is:

\[
\begin{align*}
  x_{11} &= x_{22} = 1 - y_1 & x_{12} &= x_{21} = y_1 \\
  x_{33} &= x_{44} = 1 - y_2 & x_{34} &= x_{43} = y_2 \\
  x_{55} &= x_{66} = 1 - y_3 & x_{56} &= x_{65} = y_3 \\
  x_{77} &= x_{88} = 1 - y_4 & x_{78} &= x_{87} = y_4
\end{align*}
\]

All other variables are assigned zero. This shows that $g$ can be expressed as a polynomial of degree at most 2 in the variables $y_1, y_2, y_3, y_4$. We say that $g$ has degree at most 2.

By construction, the function $g$ satisfies the following constraints:

\[
\begin{align*}
  g(0, 0, 0, 0) &= f(id) = 0 \\
  g(1, 0, 0, 0) &= f((1 2)) = 1 \\
  g(0, 1, 0, 0) &= f((3 4)) = 1 \\
  g(0, 0, 1, 0) &= f((5 6)) = 1 \\
  g(0, 0, 0, 1) &= f((7 8)) = 1
\end{align*}
\]
We say that $g$ has sensitivity 4.

There are only $2^4$ many functions from $\{0,1\}^4$ to $\{0,1\}$. For each function, we can check whether it has degree at most 2 by solving linear equations, since each function has a unique representation as a multilinear polynomial in $y_1, y_2, y_3, y_4$ (this is a basic fact in Boolean function analysis). Going over all such functions in SAGE [Sag22], we find out that none of them has sensitivity 4, a contradiction. The relevant code appears in Appendix A.1.

For experts in Boolean function analysis, here is an alternative proof that $g$ cannot have sensitivity 4. Since $\deg g \leq 2$, every influential variable has influence at least $1/2$ [OD14 Proposition 3.6]. If $g$ has sensitivity 4, then it depends on 4 variables, and so its total influence is 2. Since $g$ has degree 2, this can only happen if $g$ is homogeneous of degree 2. But in that case, the sensitivity of $g$ at every point is exactly 2 [FHKL16 Proposition 3.7].

The bound on the sensitivity cannot be improved: the Boolean function

$$g(y_1, y_2, y_3) = y_1 y_2 y_3 + (1 - y_1)(1 - y_2)(1 - y_3)$$

has degree 2 and satisfies

$$g(0,0,0) = 1, \quad g(1,0,0) = g(0,1,0) = g(0,0,1) = 0.$$ 

### 2.1.3 Degree reduction

Let $\mathcal{F} \subseteq S_n$. We denote the restriction of $\mathcal{F}$ to the coset $[i \mapsto j]$ by $\mathcal{F}|_{[i \mapsto j]}$. We can think of $\mathcal{F}|_{[i \mapsto j]}$ as a subset of $S_{n-1}$. Formally speaking, we can think of $S_n$ as the set of all injections from $[n]$ to $[n]$. The restriction of $S_n$ to the coset $[i \mapsto j]$ is isomorphic to the set of all injections from $[n] \setminus \{i\}$ to $[n] \setminus \{j\}$, which is the same as $S_{n-1}$ up to renumbering.

Let $f$ be the characteristic function of $\mathcal{F}$. The definition of degree using polynomials shows that $\deg f|_{[i \mapsto j]} \leq \deg f$, where $f$ is the characteristic function of $\mathcal{F}|_{[i \mapsto j]}$. Indeed, given a polynomial representing $f$, we can obtain a polynomial representing $f|_{[i \mapsto j]}$ by substituting $x_{ij} = 1$ and $x_{i'j'} = x_{i'j} = 0$ for any $i' \neq i$ and $j' \neq j$.

The following crucial lemma, which forms part of the proof of [DFL+21 Lemma 5.7], states that if $\mathcal{F}$ is contained in $[i \mapsto j]$, then the degree of $f$ strictly decreases when restricting to $[i \mapsto j]$.

**Lemma 2.6.** Suppose that $f: S_n \to \{0,1\}$ is the characteristic function of a family which is a subset of the coset $[i \mapsto j]$. Then $\deg f|_{[i \mapsto j]} \leq \max(\deg f - 1, 0)$.

The corresponding result for functions on the Boolean cube $\{0,1\}^n$ is easy to prove. Suppose that $f: \{0,1\}^n \to \{0,1\}$ is such that $f(x) = 1$ implies $x_n = 1$. We can write

$$f(x_1, \ldots, x_n) = (1 - x_n)f_0(x_1, \ldots, x_{n-1}) + x_n f_1(x_1, \ldots, x_{n-1}).$$

Substituting $x_n = 0$, we get $f_0 = 0$, and so $f = x_n f_1$. The restriction of $f$ to the coset $\{x : x_n = 1\}$ is $f_1$. If $f_1 \neq 0$ then clearly $\deg f = \deg f_1 + 1$, where $\deg f$ is the degree of the unique multilinear polynomial representing $f$.

The argument proving Lemma 2.6 is more subtle. The starting point is the following explicit description of the subspaces $V^\lambda$, which we mentioned in Section 2.1.1.

Let $\lambda = \lambda_1, \ldots, \lambda_m \vdash n$. A **Young tableau of shape** $\lambda$ is an arrangement of the numbers $1, \ldots, n$ in $m$ rows of lengths $\lambda_1, \ldots, \lambda_m$, left justified. Here are some examples:

$$\begin{array}{ccc}
3 & 4 & 1 \\
2 & \end{array} \quad \begin{array}{cc}
1 & 4 \\
2 & 3 \\
\end{array}$$

$$\lambda = (3,1) \quad \lambda = (2,2)$$
Let $s, t$ be a pair of Young tableaux of the same shape $\lambda_1, \ldots, \lambda_m \vdash n$. The function $e_{s,t}: S_n \to \{0, 1\}$ is defined as follows: $e_{s,t}(\alpha) = 1$ if for every $k \in [m]$, $\alpha$ sends every number on the $k$-th row of $s$ to a number on the $k$-th row of $t$, and $e_{s,t}(\alpha) = 0$ otherwise. For example, consider

$$s = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 \\
\end{array} \quad t = \begin{array}{ccc}
3 & 1 & 5 \\
2 & 4 \\
\end{array}$$

Then $e_{s,t}(\alpha) = 1$ if $\alpha(1), \alpha(2), \alpha(3) \in \{3, 1, 5\}$ and $\alpha(4), \alpha(5) \in \{2, 4\}$.

For a tableau $t$ with $m'$ columns, we denote by $C(t)$ the column stabilizer of $t$, which is the set of all permutations of the entries of $t$ which only move entries within the same column. Each permutation $\pi \in C(t)$ thus consists of $m'$ permutations $\pi_1, \ldots, \pi_{m'}$, one for each column. The sign of $\pi$, denote $(-1)^\pi$, is the product of the signs of the permutations $\pi_1, \ldots, \pi_{m'}$. We denote the result of applying $\pi$ to $t$ by $t^\pi$.

Let $s, t$ be a pair of Young tableaux of the same shape $\lambda \vdash n$. We define a function $\chi_{s,t}: S_n \to \{-1, 0, 1\}$ as follows:

$$\chi_{s,t} = \sum_{\pi \in C(t)} (-1)^\pi e_{s,t^\pi}.$$  

Continuing our above example, here is $t^\pi$ for all elements $\pi \in C(t)$, together with their sign:

$$\left( \begin{array}{ccc}
3 & 1 & 5 \\
2 & 4 \\
\end{array} \right), +1 \quad \left( \begin{array}{ccc}
3 & 4 & 5 \\
2 & 1 \\
\end{array} \right), -1 \quad \left( \begin{array}{ccc}
2 & 1 & 5 \\
3 & 4 \\
\end{array} \right), -1 \quad \left( \begin{array}{ccc}
2 & 4 & 5 \\
3 & 1 \\
\end{array} \right), +1$$

We are interested in the functions $\chi_{s,t}$ since they span $V^\lambda$.

**Theorem 2.7** ([Sag01], Chapter 2). Let $\lambda \vdash n$. The subspace $V^\lambda$ is spanned by the functions $\chi_{s,t}$, where $s, t$ go over all pairs of Young tableaux of shape $\lambda$.

We can now prove Lemma 2.6.

**Proof of Lemma 2.6.** Assume, for concreteness, that $i = j = n$. If $n = 1$ then $\deg f|_{[n \rightarrow n]} = 0$, so there is nothing to prove. If $f|_{[n \rightarrow n]} = 0$ then again $\deg f|_{[n \rightarrow n]} = 0$, so there is nothing to prove. Therefore we can assume that $n \geq 2$ and that $f|_{[n \rightarrow n]} \neq 0$.

Let $d = \deg f|_{[n \rightarrow n]}$. According to the spectral definition of degree (see Section 2.1.1), $f|_{[n \rightarrow n]}^\lambda \neq 0$ for some $\lambda \vdash n - 1$ such that $\lambda_1 = (n - 1) - d$. Since the decomposition into isotypic components is orthogonal, this implies that $\langle f|_{[n \rightarrow n]}, f|_{[n \rightarrow n]}^\lambda \rangle = \langle f|_{[n \rightarrow n]}, f|_{[n \rightarrow n]}^\lambda \rangle = 0$. Since $f|_{[n \rightarrow n]}^\mu \in V^\mu$, Theorem 2.7 implies that $\langle f|_{[n \rightarrow n]}, \chi_{s,t} \rangle = 0$ for some Young tableaux $s, t$ of shape $\mu$.

Let $s', t'$ be the Young tableaux obtained by adding a new row to $s, t$ (respectively) consisting only of the number $n$. Here is an example:

$$s = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 \\
\end{array}, \quad t = \begin{array}{ccc}
3 & 1 & 5 \\
2 & 4 \\
\end{array} \quad \rightarrow \quad s' = \begin{array}{ccc}
1 & 2 & 3 & 6 \\
4 & 5 \\
\end{array}, \quad t' = \begin{array}{ccc}
3 & 1 & 5 \\
2 & 4 & 6 \\
\end{array}$$

The tableaux $s', t'$ have shape $\mu$ satisfying $\mu_1 = \lambda_1 = (n - 1) - d = n - (d + 1)$. We will show that $\langle f, \chi_{s',t'} \rangle = 0$. The orthogonality of the decomposition into isotypic components would imply that $f^\mu \neq 0$, and so $\deg f \geq d + 1$ by the spectral definition of degree.

Let $\pi' \in C(t')$ be any permutation such that $\pi'(n) \neq n$. By assumption, $f(\alpha') = 1$ only if $\alpha'(n) = n$, in which case $e_{s',(t')}(\alpha') = 0$. Therefore $\langle f, e_{s',(t')} \rangle = 0$. This shows that

$$\langle f, \chi_{s',t'} \rangle = \sum_{\pi' \in C(t')} (-1)^{\pi'} \langle f, e_{s',(t')} \rangle = 0.$$
If \( \pi'(n) = n \) then \( e_{\pi'(t),\pi'(t')} \neq 0 \) only if \( \alpha'(n) = n \). Conversely, if \( \alpha'(n) = n \), that is, if \( \alpha' \in [n \rightarrow n] \), then \( e_{\pi'(t),\pi'(t')}(\alpha') = e_{s,t}(\alpha) \), where \( \alpha, \pi \) are the restrictions of \( \alpha', \pi' \) to \( [n-1] \). Moreover, \((-1)^\pi = (-1)^{\pi'}\). Therefore
\[
\langle f, \chi_{\pi', t'} \rangle = \sum_{\pi \in C(t)} (-1)^\pi \langle f|_{[n \rightarrow n]}, e_{s, t'} \rangle = \langle f|_{[n \rightarrow n]}, \chi_{s, t} \rangle \neq 0.
\]

\[\square\]

\section*{2.2 Perfect matching scheme}

For integer \( n \geq 1 \), the \textit{perfect matching scheme} \( \mathcal{M}_{2n} \) is the collection of all perfect matchings of the complete graph \( K_{2n} \). Since the symmetric group \( S_n \) can be viewed as the collection of all perfect matchings of the complete bipartite graph \( K_{n,n} \), we can think of the perfect matching scheme as the non-bipartite analog of the symmetric group.

The perfect matching scheme \( \mathcal{M}_{2n} \) contains \((2n-1)!!\) many different perfect matchings. Here
\[
(2n-1)!! = (2n-1)(2n-3)(2n-5) \cdots (5)(3)(1) = \frac{(2n)!}{2^{n+1}n!}.
\]

We sometimes think of a perfect matching in \( \mathcal{M}_{2n} \) as a set of \( n \) pairs of elements from \([2n]\), which together cover all of \([2n]\) (the \textit{pair representation}). At other times, it will be useful to think of a perfect matching in \( \mathcal{M}_{2n} \) as a fixed-point-free involution on \([2n]\), that is, \( m(i) \neq i \) is the element that \( i \) is matched to, and \( m(m(i)) = i \).

A \textit{t-coset} of \( \mathcal{M}_{2n} \) is a set of the form
\[
\{ m \in \mathcal{M}_{2n} : m(i_1) = j_1, \ldots, m(i_t) = j_t \},
\]
where \( i_1, \ldots, i_t, j_1, \ldots, j_t \in [2n] \) are distinct\(^1\) A t-coset contains \((2(n-t)-1)!!\) perfect matchings.

We use the term \textit{coset} for a 1-coset. For \( i \neq j \in [2n] \), the coset \([i \leftrightarrow j]\) consists of all perfect matchings in which \( i, j \) are matched:
\[
[i \leftrightarrow j] = \{ m \in \mathcal{M}_{2n} : m(i) = j \}.
\]

Note that \([i \leftrightarrow j] = [j \leftrightarrow i] \).

A subset \( \mathcal{F} \subseteq \mathcal{M}_{2n} \) is \textit{t-intersecting} if any two perfect matchings in \( \mathcal{F} \) are \( t \)-intersecting.

\subsection*{2.2.1 Degree}

We can represent perfect matchings \( m \in \mathcal{M}_{2n} \) using \( \binom{n}{2} \) variables \( x_{ij} \) (where the index is an unordered pair of elements) whose semantics are: \( x_{ij} = 1 \) if \( m(i) = j \), and \( x_{ij} = 0 \) otherwise. The \textit{degree} of a function \( f : \mathcal{M}_{2n} \rightarrow \mathbb{R} \), denoted \( \text{deg } f \), is the minimal \( d \) such that \( f \) can be represented as a degree \( d \) polynomial over the variables \( x_{ij} \). For example, the characteristic function of a \textit{t-coset} has degree at most \( t \).

An equivalent way to define degree is via the representation theory of the perfect matching scheme. Let \( \mathbb{R}[\mathcal{M}_{2n}] \) be the vector space of all real-valued functions over the perfect matching scheme. Representation theory (for example, \cite[Chapter 11]{CSST05}) gives an orthogonal decomposition (with respect to the inner product \( \langle f, g \rangle = \sum_{m \in \mathcal{M}_{2n}} f(m)g(m) \))
\[
\mathbb{R}[\mathcal{M}_{2n}] = \bigoplus_{\lambda=n} V^{2\lambda},
\]
where \( \lambda \) goes over all integer partitions of \( n \), \( 2\lambda \) is the partition of \( 2n \) obtained by doubling each part, and \( V^{2\lambda} \) are the isotypic components, which we define explicitly in Section 2.2.3\(^1\) Accordingly, every function \( f : \mathcal{M}_{2n} \rightarrow \mathbb{R} \) has a unique decomposition
\[
f = \sum_{\lambda=n} f^{=\lambda}.
\]

\(^1\)This is a slight abuse of terminology as \( \mathcal{M}_{2n} \) doesn’t afford a natural group structure.
where \( f^\lambda \in V^{2^\lambda} \). Lindzey [Lin18b] Theorem 5.1.1 showed that \( \deg f \) is the maximal \( d \) such that \( f^\lambda \neq 0 \) for some \( \lambda \) satisfying \( \lambda_1 = n - d \). In particular, every function has degree at most \( n - 1 \).

We can now formally state the main result of Fallat, Meagher and Shirazi [FMS21] Theorem 4.13.

**Theorem 2.8.** If \( n \geq 3 \) then any 2-intersecting family \( \mathcal{F} \subseteq \mathcal{M}_{2n} \) contains at most \((2n-5)!!\) perfect matchings. Furthermore, if \( f : \mathcal{M}_{2n} \rightarrow \{0,1\} \) is the characteristic vector of a 2-intersecting family of size \((2n-5)!!\), then \( \deg f \leq 2 \).

We will also need a result of Dafni et al. [DFL+21] classifying degree 1 functions.

**Theorem 2.9** ([DFL+21] Theorem 6.2). If \( f : \mathcal{M}_{2n} \rightarrow \{0,1\} \) has degree at most 1 then \( f \) is either a dictator:

\[
f = \sum_{j \in J} x_{ij},
\]

for some \( i \in [2n] \) and \( J \subseteq [2n] \) not containing \( i \); or \( f \) is a triangle:

\[
f = x_{ij} + x_{ik} + x_{jk},
\]

for some distinct \( i, j, k \in [2n] \); or \( f \) is an anti-triangle:

\[
f = 1 - x_{ij} - x_{ik} - x_{jk},
\]

for some distinct \( i, j, k \in [2n] \).

### 2.2.2 Certificate complexity

Fix \( n \). A **certificate** is a set \( C = \{ \{i_1, j_1\}, \ldots, \{i_r, j_r\} \} \subseteq \binom{[2n]}{2} \), where \( \binom{[2n]}{2} \) is the collection of all subsets of \([2n]\) of size 2. A perfect matching \( m \in \mathcal{M}_{2n} \) satisfies the certificate \( C \) if \( m(i_1) = j_1, \ldots, m(i_r) = j_r \).

A **Boolean function** \( f \) is a function \( f : \mathcal{M}_{2n} \rightarrow \{0,1\} \). Just as in Section 2.1.2 for each \( m \in \mathcal{M}_{2n} \), we can define the following notions: a certificate for \( m \) (with respect to \( f \)); the certificate complexity of \( m \), denoted \( C(f, m) \); a minimum certificate; and the certificate complexity of \( f \), denoted \( C(f) \). As in the case of the symmetric group, \( C(f) \leq n - 1 \), since a perfect matching is determined by any \( n - 1 \) of its edges. Dafni et al. [DFL+21] showed that \( C(f) \) is polynomially related to \( \deg f \).

In the case of the symmetric group, degree 1 functions have certificate complexity 1. This is no longer the case for the perfect matching scheme, since triangles and anti-triangles have certificate complexity 2.

**Lemma 2.10.** If \( f : \mathcal{M}_{2n} \rightarrow \{0,1\} \) has degree at most 1 then \( C(f) \leq 2 \).

**Proof.** According to Theorem 2.9, \( f \) is either a dictator, a triangle, or an anti-triangle. If \( f = \sum_{j \in J} x_{ij} \) is a dictator, then every \( m \) has a certificate \( \{\{i, m(i)\}\} \). If \( f = x_{ij} + x_{ik} + x_{jk} \) is a triangle, then every \( m \) such that \( f(m) = 1 \) has one of the certificates \( \{\{i, j\}\}, \{\{i, k\}\}, \{\{j, k\}\} \), and every \( m \) such that \( f(m) = 0 \) has the certificate \( \{\{i, m(i)\}, \{j, m(j)\}\} \). If \( f \) is an anti-triangle then \( 1 - f \) is a triangle, so we can use the same certificates as in the case of a triangle. \( \square \)

The proof of Lemma 2.4 extends to the perfect matching scheme with minor changes.

**Lemma 2.11.** Let \( f : \mathcal{M}_{2n} \rightarrow \{0,1\} \) be the characteristic function of a 2-intersecting family, and suppose that \( C(f) \leq n - 2 \). If \( f(m_1) = f(m_2) = 1 \) and \( C_{m_1}, C_{m_2} \) are minimum certificates for \( m_1, m_2 \) (respectively), then \( |C_{m_1} \cap C_{m_2}| \geq 2 \).

**Proof.** We will show how to construct a perfect matching \( m_1' \) satisfying \( C_{m_1} \) such that \( m_1' \cap C_{m_2} = C_{m_1} \cap C_{m_2} \).

The same argument can be used to construct a perfect matching \( m_2' \) satisfying \( C_{m_2} \) such that \( m_1' \cap m_2' = m_1' \cap C_{m_2} \). It follows that \( |C_{m_1} \cap C_{m_2}| = |m_1' \cap m_2'| \geq 2 \), since \( f \) is the characteristic function of a 2-intersecting family.
Let \( C_{m_1} = \{i_1, j_1, \ldots, i_r, j_r\} \), where \( r \leq n - 2 \), and let \( k_1, \ldots, k_{2(n-r)} \) be the indices not mentioned in \( C_{m_1} \). Rearrange the indices so that if \( \{k_i, k_j\} \subseteq C_{m_2} \) then \( \{s, t\} = \{2t - 1, 2t\} \) for some \( t \in [n-r] \). The perfect matching \( m'_1 \) has the pair representation

\[
\{i_1, j_1\}, \ldots, \{i_r, j_r\}, \{k_1, k_{n-r+1}\}, \{k_2, k_{n-r+2}\}, \ldots, \{k_{n-r}, k_{2(n-r)}\}.
\]

Since \( n - r \geq 2 \), the new pairs do not intersect \( C_{m_2} \), completing the proof. \( \square \)

Using sensitivity, we can rule out the case \( C(f) = n - 1 \) for functions of degree at most 2, just as we did for the symmetric group.

**Lemma 2.12.** If \( n \geq 8 \) and \( f: M_{2n} \to \{0, 1\} \) has degree at most 2, then \( C(f) \leq n - 2 \).

**Proof.** The proof is by contradiction. Suppose that \( n \geq 8 \), that \( f: M_{2n} \to \{0, 1\} \) has degree at most 2, and that \( C(f) = n - 1 \), say \( C(f, m) = n - 1 \) for \( m = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}\} \). Since \( C(f, m) > n - 2 \), \( m \setminus \{\{2i - 1, 2i\}, \{2j - 1, 2j\}\} \) is not a certificate for \( m \) for any \( i \neq j \in [n] \), and so \( f(m^i) \neq f(m) \), where \( m^i \) is obtained from \( m \) by replacing \( \{\{2i - 1, 2i\}, \{2j - 1, 2j\}\} \) with \( \{\{2i - 1, 2j\}, \{2j - 1, 2i\}\} \).

We construct a function \( g: \{0, 1\}^4 \to \{0, 1\} \) as follows: \( g(y_1, y_2, y_3, y_4) \) is the value of \( f \) on the perfect matching obtained from \( m \) by applying the subset of the operations \( 12, 34, 56, 78 \) specified by the input (for example, if \( y_1 = 1 \) then we apply \( 12 \)). As in the proof of Lemma 2.5, this results in a function of degree at most 2 satisfying \( g(0, 0, 0, 0) \neq g(1, 0, 0, 0) = g(0, 1, 0, 0) = g(0, 0, 1, 0) = g(0, 0, 0, 1) \), which is impossible. \( \square \)

2.2.3 Degree reduction

Let \( F \subseteq M_{2n} \). We denote the restriction of \( F \) to the coset \([i \leftrightarrow j]\) by \( F|_{[i \leftrightarrow j]} \), which we can think of as a subset of \( M_{2(n-1)} \).

Let \( f \) be the characteristic function of \( F \). The definition of degree using polynomials shows that \( \text{deg } f|_{[i \leftrightarrow j]} \leq \text{deg } f \). The following analog of Lemma 2.6 shows that if \( F \) is contained in \([i \leftrightarrow j]\), then the degree of \( f \) strictly decreases when restriction to \([i \leftrightarrow j]\).

**Lemma 2.13.** Suppose that \( f: M_{2n} \to \{0, 1\} \) is the characteristic function of a family which is a subset of the coset \([i \leftrightarrow j]\). Then \( \text{deg } f|_{[i \leftrightarrow j]} \leq \max(\text{deg } f - 1, 0) \).

Before we can prove this lemma, we need an explicit description of the subspaces \( V^{2\lambda} \), which were mentioned in Section 2.2.1.

Let \( \lambda \vdash n \), and let \( t \) be a Young tableau of shape \( 2\lambda \). The function \( e_t: M_{2n} \to \{0, 1\} \) is defined as follows: \( e_t(m) = 1 \) if for all \( i \in [2n] \), the element \( m(i) \) is in the same row of \( t \) as \( i \), and \( e_t(m) = 0 \) otherwise. For example, consider

\[
t = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}
\]

Then \( e_t(m) = 1 \) if \( m(1), m(2), m(3), m(4) \in \{1, 2, 3, 4\} \) and \( m(5), m(6), m(7), m(8) \in \{5, 6, 7, 8\} \).

Given a Young tableau \( t \) of shape \( 2\lambda \), we define a function \( \chi_t: M_{2n} \to \{-1, 0, 1\} \) as follows:

\[
\chi_t = \sum_{\pi \in C(t)} (-1)^s e_{t^\pi}.
\]

These functions span \( V^{2\lambda} \).

**Theorem 2.14** ([Lin18b, Theorem 5.2.6]). Let \( \lambda \vdash n \). The subspace \( V^{2\lambda} \) is spanned by the functions \( \chi_t \), where \( t \) goes over all Young tableaux of shape \( 2\lambda \).

We can now prove Lemma 2.13. The proof is along the lines of Lemma 2.6.
Proof of Lemma 2.13. Assume, for concreteness, that $i = 2n - 1$ and $j = 2n$. If $n = 1$ then $\deg f|_{[2n-1 \leftrightarrow 2n]} = 0$, so there is nothing to prove. Similarly, if $f|_{[2n-1 \leftrightarrow 2n]} = 0$ then there is nothing to prove. Therefore we can assume that $n \geq 2$ and that $f|_{[2n-1 \leftrightarrow 2n]} \neq 0$.

Let $d = \deg f|_{[2n-1 \leftrightarrow 2n]}$. According to the spectral definition of degree (see Section 2.2.1), $f|_{[2n-1 \leftrightarrow 2n]} \neq 0$ for some $\lambda \vdash n - 1$ such that $\lambda_1 = (n - 1) - d$. Since the decomposition into isotypic components is orthogonal, Theorem 2.14 implies that $\langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_t \rangle \neq 0$ for some Young tableau $t$ of shape $\lambda$. Among all tableau $t$ with $2(n - 1 - d)$ squares on the first lines satisfying $\langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_t \rangle \neq 0$, we choose one which maximizes the number of rows (which is the number of parts in the corresponding partition), say $t$ has $R$ rows.

Let $t'$ be the Young tableau obtained by adding a new row to $t$ consisting only of the numbers $2n - 1, 2n$. The tableau $t'$ has shape $\mu$ satisfying $\mu_1 = \lambda_1 = n - (d + 1)$. We will show that $\langle f, \chi_{t'} \rangle \neq 0$, and so $\deg f \geq d + 1$ by the spectral definition of degree.

If $\pi' \in C(t')$ is any permutation such that $2n - 1, 2n$ do not end up in the same row, then $\langle f, e(\pi') \rangle = 0$, since $e(\pi') \neq (m) \neq 0$ implies that $m(2n - 1) \neq 2n$, and so $f(m) = 0$. Therefore

$$\langle f, \chi_{t'} \rangle = \sum_{r=1}^{R+1} \sum_{\pi' \in C(t')} (-1)^{\pi'} \langle f, e(\pi') \rangle.$$ 

For $r \leq R$, let $t''$ be the tableau obtained from $t$ by taking the first two elements of row $r$, moving them to a new row at then very end, removing row $r$ if it is empty, and otherwise sliding the remainder of row $r$ to the left. For example:

\begin{align*}
t &= \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{array} & t_1 &= \begin{array}{cccc}
3 & 4 & 5 & 6 \\
7 & 8 & 1 & 2 \\
\end{array} & t_2 &= \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{array} & t^3 &= t
\end{align*}

We also define $t^{R+1} = t$. Observe that

$$\langle f, \chi_{t'} \rangle = \sum_{r=1}^{R+1} \sum_{\pi' \in C(t')} (-1)^{\pi'} \langle f|_{[2n-1 \leftrightarrow 2n]}, e(\pi') \rangle = \sum_r \langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_{t'} \rangle.$$ 

If the $r$-th row of $t$ contains more than two elements then $t'$ contains more rows than $t$, and so by the choice of $t$, we have $\langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_{t'} \rangle = 0$. Otherwise, $\chi_{t'}$ differs from $\chi_t$ by an identical permutation of the first two columns, and so $\langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_{t'} \rangle = \langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_t \rangle$. Therefore, denoting by $\ell$ the number of rows of $t$ of length 2, we have

$$\langle f, \chi_{t'} \rangle = (\ell + 1)\langle f|_{[2n-1 \leftrightarrow 2n]}, \chi_t \rangle \neq 0.$$ 

The reason we get $\ell + 1$ rather than $\ell$ is the choice $r = R + 1$, which always leads to $t^{R+1} = t$. 

3 Main theorem: symmetric group

3.1 Structure theorem

We start with a simple observation on 2-intersecting families, whose statement requires a few definitions.

Two cosets $[i_1 \rightarrow j_1], [i_2 \rightarrow j_2]$ are compatible if their intersection is non-empty. In other words, either the two cosets are equal, or $i_1 \neq i_2$ and $j_1 \neq j_2$. A family $\mathcal{F} \subseteq S_n$ is $m$-covered, for some integer $m \geq 1$, if there are $m$ compatible cosets $[i_1 \rightarrow j_1], \ldots, [i_m \rightarrow j_m]$ such that $\mathcal{F} \subseteq [i_1 \rightarrow j_1] \cup \cdots \cup [i_m \rightarrow j_m]$.

Lemma 3.1. Let $f$ be the characteristic function of a 2-intersecting family $\mathcal{F} \subseteq S_n$, and suppose that $C(f) \leq n - 2$.

If $\alpha \in \mathcal{F}$ then $\mathcal{F}$ is $m$-covered for $m = C(f, \alpha) - 1$. 

12
Moreover, we are now ready to prove our main upper bound, from which Theorem 1.3 will quickly follow. We showed above that the restriction of \( f \) to \([i_1 \mapsto j_1] \cup \cdots \cup [i_m \mapsto j_m]\). Let \( \alpha \) be a minimal certificate for \( \beta \). According to Lemma 2.6 \( |C_\alpha \cap C_\beta| \geq 2 \). Therefore \((i_s, j_s) \in C_\beta\) for some \( s \in [m] \), implying that \( \beta \in [i_s \mapsto j_s] \subseteq [i_1 \mapsto j_1] \cup \cdots \cup [i_m \mapsto j_m] \).

The following structure lemma provides a kind of converse to Lemma 3.1.

**Lemma 3.2.** Let \( \mathcal{F} \subseteq S_n \) be an \( m \)-covered family, say \( \mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \cdots \cup [i_m \mapsto j_m] \). Suppose that its characteristic function \( f \) has degree at most 2.

Every \( \alpha \) has a certificate of the form \( \{(i_1, \alpha(i_1)), \ldots, (i_m, \alpha(i_m)), (i, j)\} \), and in particular, \( C(f) \leq m + 1 \).

**Proof.** We will prove by induction on \( m \) that if \( \mathcal{F} \subseteq S_n \) is \( m \)-covered, say \( \mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \cdots \cup [i_m \mapsto j_m] \), and its characteristic function \( f \) has degree at most 2, then for every distinct \( k_1, \ldots, k_m \in [n] \), the restriction of \( f \) to \([i_1 \mapsto k_1] \cap \cdots \cap [i_m \mapsto k_m]\) has degree at most 1. By Lemma 2.3 this restriction has certificate complexity at most 1. For any \( \alpha \in S_n \), choosing \( k_1 = \alpha(i_1), \ldots, k_m = \alpha(i_m) \), we deduce that \( \alpha \) has a certificate of the required form.

We start with the base case \( m = 1 \). Let \( \mathcal{F} \subseteq S_n \) be such that \( \mathcal{F} \subseteq [i_1 \mapsto j_1] \), and suppose that its characteristic function \( f \) has degree at most 2. Lemma 2.6 directly implies that the restriction of \( f \) to \([i_1 \mapsto j_1]\) has degree at most 1, as required.

Suppose now that \( m \geq 2 \). Let \( \mathcal{F} \subseteq S_n \) be such that \( \mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \cdots \cup [i_m \mapsto j_m] \), and suppose that its characteristic function \( f \) has degree at most 2. Consider any distinct \( k_1, \ldots, k_m \in [n] \). We need to show that the restriction of \( f \) to \([i_1 \mapsto k_1] \cap \cdots \cap [i_m \mapsto k_m]\) has degree at most 1.

Suppose first that \( k_s \neq j_s \) for some \( s \in [m] \), say \( k_m \neq j_m \). The restriction of \( \mathcal{F} \) to \([i_m \mapsto k_m]\) is contained in \([i_1 \mapsto j_1] \cup \cdots \cup [i_{m-1} \mapsto j_{m-1}]\), and its characteristic function has degree at most 2. Therefore the induction hypothesis implies that the restriction of \( f \) to \([i_m \mapsto k_m]\) has degree at most 1.

It remains to show that the restriction of \( f \) to \([i_1 \mapsto j_1] \cap \cdots \cap [i_m \mapsto j_m]\) has degree at most 1. To see this, consider the function

\[
g := x_{i_1 j_1} \cdots x_{i_m j_m} f = f - \sum_{(k_1, \ldots, k_m) \neq (j_1, \ldots, j_m)} x_{i_1 k_1} \cdots x_{i_m k_m} f.
\]

We showed above that the restriction of \( f \) to \([i_1 \mapsto k_1] \cap \cdots \cap [i_m \mapsto k_m]\) has degree at most 1 whenever \((k_1, \ldots, k_m) \neq (j_1, \ldots, j_m)\). Therefore every term in the sum on the right-hand side has degree at most \( m + 1 \), since it can be represented as \( x_{i_1 k_1} \cdots x_{i_m k_m} \) times the linear polynomial representing the restriction of \( f \) to \([i_1 \mapsto k_1] \cap \cdots \cap [i_m \mapsto k_m]\). Consequently, \( \deg g \leq \max(\deg f, m + 1) = m + 1 \).

By construction, \( g \) is the characteristic function of a family which is contained in the coset intersection \( \mathcal{I} = [i_1 \mapsto j_1] \cap \cdots \cap [i_m \mapsto j_m] \). Applying Lemma 2.6 repeatedly, we conclude that the restriction of \( g \) to \( \mathcal{I} \) has degree at most \( \max(m + 1 - m, 0) = 1 \). Since the restriction of \( f \) to \( \mathcal{I} \) is the same as the restriction of \( g \) to \( \mathcal{I} \), this completes the proof of the induction step.

**3.2 Proof of main theorem**

We are now ready to prove our main upper bound, from which Theorem 1.3 will quickly follow.

**Lemma 3.3.** Let \( \mathcal{F} \subseteq S_n \) be a \( 2 \)-intersecting family whose characteristic function \( f \) has degree at most 2. Suppose that \( C(f) \leq n - 2 \).

The family \( \mathcal{F} \) is contained in the union of at most \( T \) many \( C(f) \)-cosets, where

\[
T = \frac{2|C(f)/2|(|C(f)| - 1)!}{2|C(f)/2|}.
\]

Moreover, \( C(f) \geq 2 \).
Proof. If $\mathcal{F}$ is empty then the lemma is vacuously true. Otherwise, choose an arbitrary $\beta \in \mathcal{F}$. Lemma 2.4 applied to $\alpha = \beta$, shows that $C(f, \beta) \geq 2$. Lemma 3.1 shows that $\mathcal{F}$ is $m$-covered, where $m = C(f, \beta) - 1$. Without loss of generality, we can assume that $\mathcal{F} \subseteq \{1 \mapsto 1\} \cup \cdots \cup \{m \mapsto m\}$.

Lemma 3.2 shows that $C(f) \leq m + 1$, and so $m = C(f) - 1$ and $C(f) = C(f, \beta) \geq 2$. The lemma also shows that every $\alpha \in \mathcal{F}$ has a certificate $C_\alpha$ of the form $\{(1, \alpha(1)), \ldots, (m, \alpha(m)), (i, j)\}$. It also implies that $\mathcal{F}$ is not $(m-1)$-covered (otherwise its certificate complexity would be at most $m < C(f)$). Therefore, if $X$ is any set of at most $m-1$ compatible cosets, then there is $\alpha_\mathcal{X} \in \mathcal{F}$ which is not contained in the union of the cosets in $X$. Fix a certificate $C_X$ for $\alpha_\mathcal{X}$ of the form $\{(1, \alpha_X(1)), \ldots, (m, \alpha_X(m)), (i_X, j_X)\}$.

We think of the certificates $C_\alpha, C_X$ as colored using $m + 1$ colors: for $k \in [m]$, a pair $(k, \ast)$ is colored $k$, and the remaining pair $(i, j)$ is colored $m + 1$. We can determine the color of a pair $(i', j')$ given only the pair: if $i' \in [m]$ then the color is $i'$, and otherwise the color is $m + 1$.

We will prove the lemma by showing that if $\alpha \in \mathcal{F}$ then $C_\alpha$ is one of at most $T$ possible certificates. Since $C_\alpha$ corresponds to a $C(f)$-coset containing $\alpha$, the lemma would follow. In the proof, we will freely identify a pair $(i, j)$ with the corresponding coset $[i \mapsto j]$, and from now on we will only ever mention cosets.

The proof is slightly different depending on the parity of $C(f)$. We start with the case in which $C(f) = 2r$ is even. Let $\alpha \in \mathcal{F}$. We define a sequence $X_0 \subset \cdots \subset X_r \subseteq C_\alpha$, where $X_s$ is a collection of $2s$ compatible cosets, as follows. The starting point is $X_0 = \emptyset$. Now let $s < r$, and suppose that $X_s$ has been defined. Since $|X_s| = 2s \leq C(f) - 2 = m - 1$, there is a permutation $\alpha_X \in \mathcal{F}$ which lies outside of the union of the cosets in $X_s$. According to Lemma 2.4 the certificates $C_\alpha$ and $C_X$ must have at least two cosets in common $q_1, q_2$. The choice of $\alpha_X$ guarantees that $q_1, q_2 \notin X_s$, and we define $X_{s+1} = X_s \cup \{q_1, q_2\}$. By construction, $X_{s+1} \subseteq C_\alpha$, and so the cosets in $X_{s+1}$ are indeed compatible.

Note that $|X_r| = 2r = |C_\alpha|$, and so $X_r = C_\alpha$. For $s < r$, given $X_s$, the set $X_{s+1} \setminus X_s$ consists of two cosets belonging to $C_X$. Moreover, since $X_{s+1} \subseteq C_\alpha$ and $\alpha_X$ contains one coset of each color, the colors of the two cosets in $X_{s+1} \setminus X_s$ are different from the colors of the cosets in $X_s$. Since $C_X$ also contains one coset of each color, this means that there are $\binom{|C_X| - |X_s|}{2} = \binom{C(f) - 2s}{2}$ choices for $X_{s+1} \setminus X_s$. In total, the number of possible choices for $C_\alpha$ is
\[
\prod_{s=0}^{r-1} \binom{C(f) - 2s}{2} = \frac{C(f)!}{2^r},
\]
moving the formula for $T$ in the statement of the lemma.

Now suppose that $C(f) = 2r + 1$ is odd. Let $\alpha \in \mathcal{F}$. We define a sequence $X_0 \subset \cdots \subset X_r \subseteq C_\alpha$, where $X_s$ is a collection of $2s + 1$ compatible cosets, as follows. We start by defining $X_0$. According to Lemma 2.4 the certificates $C_\alpha$ and $C_\emptyset$ must contain at least two cosets in common $q_1, q_2$. At least one of these cosets is of the form $(k, \alpha(k))$ for some $k \in [m]$. We define $X_0 = \{(k, \alpha(k))\}$. Now let $s < r$, and suppose that $X_s$ has been defined. Since $|X_s| = 2s + 1 \leq C(f) - 2 = m - 1$, we can define $X_{s+1}$ as in the case of even $C(f)$.

Note that $|X_r| = 2r + 1 = |C_\alpha|$, and so $X_r = C_\alpha$. The certificate $X_0$ contains a single coset $(k, \ell) \in C_\emptyset$, where $k \in [m]$, and so there are $m = C(f) - 1$ choices for $X_0$. For $s < r$, given $X_s$, there are $\binom{|C_X| - |X_s|}{2} = \binom{C(f) - 2s}{2}$ choices for $X_{s+1}$ (this is the same as in the case of even $C(f)$). In total, the number of possible choices for $C_\alpha$ is
\[
(C(f) - 1) \prod_{s=0}^{r-1} \binom{C(f) - 2s - 1}{2} = (C(f) - 1) \frac{(C(f) - 1)!}{2^r},
\]
moving the formula for $T$ in the statement of the lemma. 

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Let $\mathcal{F}$ be a 2-intersecting subset of $S_n$ of size $(n-2)!$, where $n \geq 2$, and let $f$ be its characteristic function. Our goal is to show that
\[
\mathcal{F} = \{\alpha \in S_n : \alpha(i_1) = j_1 \text{ and } \alpha(i_2) = j_2\}
\]
for some \( i_1 \neq i_2 \in [n] \) and \( j_1 \neq j_2 \in [n] \). In other words, we need to show that \( \mathcal{F} \) is a 2-coset.

Suppose first that \( n \geq 8 \). Since \( n \geq 5 \), Theorem 2.1 shows that \( \text{deg } f \leq 2 \). Lemma 2.5 shows that \( C(f) \leq n - 2 \). Lemma 3.3 implies that \( C(f) \geq 2 \) and \( |\mathcal{F}| \leq \frac{T(n - C(f))!}{[C(f)/2]!(C(f) - 1)!/2^{C(f)/2}} \), where \( T = [C(f)/2]![C(f) - 1]/2^{C(f)/2} \).

Therefore
\[
\frac{(n - 2)!}{(n - C(f))!} \leq \frac{2[C(f)/2]!(C(f) - 1)!}{2^{C(f)/2}}.
\]

The left-hand side is clearly increasing in \( n \). Since \( n \geq C(f) + 2 \), it follows that
\[
\frac{C(f)!}{2} \leq \frac{2[C(f)/2]!(C(f) - 1)!}{2^{C(f)/2}},
\]

and so
\[
C(f) \leq \frac{2[C(f)/2]!}{2^{C(f)/2} - 1}.
\]

If \( C(f) = 2r \) is even then this reads \( 2r \leq 2r/2^{r-1} \), and so \( r = 1 \). If \( C(f) = 2r + 1 \) is odd then this reads \( 2r + 1 \leq 2r/2^{r-1} \), which never holds. We conclude that \( C(f) = 2 \).

Let \( \alpha \in \mathcal{F} \) be arbitrary. Since \( C(f) = 2 \), there is a certificate of the form \( \{\alpha(i_1), \alpha(i_2), \alpha(j_2)\} \) for \( \alpha \). The number of permutations satisfying this certificate is \((n - 2)!\), and we conclude that \( \mathcal{F} \) consists of all permutations satisfying the certificate, completing the proof in this case.

In order to complete the proof, we consider the case \( n \leq 7 \). It suffices to show that for \( n \in \{2, 3, 4, 5, 6, 7\} \), any 2-intersecting family of size \((n - 2)!\) which contains the identity permutation is of the form
\[
\{\alpha \in S_n : \alpha(i_1) = i_1, \alpha(i_2) = i_2\}
\]
for some \( i_1 \neq i_2 \).

If \( n = 2 \) or \( n = 3 \) then \((n - 2)! = 1\), and the claim can be verified directly. In order to verify the remaining cases \( n \in \{4, 5, 6, 7\} \), we formulate the task as a maximum clique problem. Let \( G_n \) be the graph whose vertex set consists of all permutations which 2-intersect the identity permutation, and in which two permutations are connected if they 2-intersect. A 2-intersecting family containing the identity permutation is the same as a clique in \( G_n \). Using SAGE [Sag22], which internally uses the software library Cliquer [NO03], we verify that a maximum clique in \( G_n \) contains \((n - 2)!\) many permutations, and furthermore, there are exactly \( \binom{n}{2} \) many maximum cliques, matching the number of 2-cosets containing the identity permutation. The relevant code appears in Appendix A.2. This completes the proof.

\[\square\]

4 Main theorem: perfect matching scheme

The proof of Theorem 1.4 is similar to that of Theorem 1.3 but there is an additional complication: Lemma 3.10 only states that degree 1 Boolean functions have certificate complexity 2. This causes the proof of Lemma 4.2 to fail. In order to get around this, we show that the bad case never occurs for (inclusion-)maximal 2-intersecting families.

Fix \( n \geq 1 \), and work over \( \mathcal{M}_{2n} \). We start by defining extended certificates. An extended certificate of size \( c \) is either a standard certificate of size \( c \), or a pair \((C', \{i, j, k\})\), where \( C' \) is a certificate of size \( c - 1 \) and \( i, j, k \in [2n] \) are distinct elements not appearing in \( C' \). A perfect matching \( m \in \mathcal{M}_{2n} \) satisfies a certificate \((C', \{i, j, k\})\) if it satisfies \( C' \) and doesn’t contain any of the pairs \( \{i, j\}, \{i, k\}, \{j, k\} \).

**Lemma 4.1.** Fix \( n \geq 1 \), work over \( \mathcal{M}_{2n} \), and let \( c \leq n - 2 \). Let \( C_1 = (C'_1, \{i, j, k\}) \) be an extended certificate of size \( c \), and let \( C_2 \) be an extended certificate of size \( c \). If any perfect matching satisfying \( C_1 \) and any perfect matching satisfying \( C_2 \) are 2-intersecting, then the same holds when \( C_1 \) is replaced by \( C'_1 \).

**Proof.** Suppose that \( m_1 \) is a perfect matching satisfying \( C'_1 \) but not \( C_1 \). Without loss of generality, \( m_1(i) = j \). We need to show that if \( m_2 \) is a perfect matching satisfying \( C_2 \) then \( m_1 \) and \( m_2 \) are 2-intersecting.
Since \( m_1 \) satisfies \( C_1 \), its pair representation consists of \( C_1 \) together with \( n - (c - 1) \geq 3 \) more pairs, one of which is \( \{i, j\} \). Among the other pairs, at least one \( \{a, b\} \) does not involve \( k \). Consider the following two perfect matchings:

\[
\begin{align*}
m'_1 &= m_1 \setminus \{\{i, j\}, \{a, b\}\} \cup \{\{i, a\}, \{j, b\}\}, \\
m''_1 &= m_1 \setminus \{\{i, j\}, \{a, b\}\} \cup \{\{i, b\}, \{j, a\}\}.
\end{align*}
\]

Both of these perfect matchings avoid the edges \( \{i, j\}, \{i, k\}, \{j, k\} \). If \( m_2 \) contains \( \{i, a\} \) then it cannot contain \( \{i, b\} \) or \( \{j, a\} \), and similarly for \( \{j, b\}, \{i, b\}, \{j, a\} \). Therefore \( m_2 \) cannot intersect both \( \{\{i, a\}, \{j, b\}\} \) and \( \{\{i, b\}, \{j, a\}\} \). Suppose that \( m_2 \) does not intersect \( \{\{i, a\}, \{j, b\}\} \). Then \( m'_1 \cap m_2 \subseteq m_1 \cap m_2 \). On the other hand, \( m'_1 \) satisfies \( C_1 \), and so \( |m'_1 \cap m_2| \geq 2 \). We conclude that \( |m_1 \cap m_2| \geq 2 \), as needed. \( \square \)

### 4.1 Structure theorem

Two cosets \([i_1 \leftrightarrow j_1], [i_2 \leftrightarrow j_2]\) are compatible if their intersection is non-empty. A family is \( r \)-covered if it is contained in the union of \( r \) compatible cosets.

**Lemma 4.2.** Let \( f \) be the characteristic function of a 2-intersecting family \( F \subseteq M_{2n} \), and suppose that \( C(f) \leq n - 2 \). If \( m \in F \) then \( F \) is \( r \)-covered for \( r = C(f, m) - 1 \).

Since the proof is identical to that of Lemma \( 3.1 \) (with Lemma \( 2.11 \) standing for Lemma \( 2.4 \)), we omit it.

The following structure theorem is the analog of Lemma \( 3.2 \). There are three differences in the statement. First, we assume that \( F \) is maximal: it is not properly contained in any larger 2-intersecting family. Second, we assume that \( r \leq n - 2 \). Third, we only provide certificates to perfect matchings which belong to \( F \).

**Lemma 4.3.** Let \( F \subseteq M_{2n} \) be an \( r \)-covered family, say \( F \subseteq [i_1 \leftrightarrow j_1] \cup \cdots \cup [i_r \leftrightarrow j_r] \), where \( r \leq n - 3 \). Suppose that \( F \) is maximal, and that its characteristic function \( f \) has degree at most 2.

Every \( m \in F \) has a certificate of the form \( \{\{i_1, m(i_1)\}, \ldots, \{i_r, m(i_r)\}, \{i, j\}\} \).

Note that the certificate could mention the same pair twice, for example if \( m(i_1) = i_2 \).

**Proof.** The first step is to prove that for every distinct \( k_1, \ldots, k_r \), the restriction of \( f \) to \([i_1 \leftrightarrow k_1] \cap \cdots \cap [i_r \leftrightarrow k_r]\) has degree at most 1, assuming the coset intersection is non-empty. This part is identical to the proof of Lemma \( 3.2 \) so we do not repeat it here (the only difference is that we never consider \( k_1, \ldots, k_r \) such that the coset intersection is empty).

Now suppose that \( m \in F \). Then the restriction of \( f \) to \( I = [i_1 \leftrightarrow m(i_1)] \cap \cdots \cap [i_r \leftrightarrow m(i_r)] \) (which is non-empty) has degree 1. According to Theorem \( 2.9 \), \( f|_I \) is either a dictator, a triangle, or an anti-triangle. If \( f|_I \) is a dictator or a triangle, then \( m|_I \) has a certificate of size 1, and we are done. We would like to show that the remaining case is impossible.

Suppose that \( f|_I \) is an anti-triangle. Then \( m \) has an extended certificate \( (C', \{i, j, k\}) \) of size \( r + 1 \leq n - 2 \). The same argument shows that every perfect matching in \( F \) has an extended certificate of size \( r + 1 \leq n - 2 \). Therefore Lemma \( 4.1 \) implies that if \( m' \notin F \) is a perfect matching satisfying \( C' \), then \( F \cup \{m'\} \) is also 2-intersecting. Hence either \( F \) is not maximal (if such \( m' \notin F \) exists), or in fact \( C' \) is already a certificate for \( m \). \( \square \)

### 4.2 Proof of main theorem

We can now prove the analog of Lemma \( 3.3 \) from which Theorem \( 1.4 \) will easily follow. The only difference in the statement is the additional assumption that \( F \) is maximal.

**Lemma 4.4.** Let \( F \subseteq M_{2n} \) be a maximal 2-intersecting family whose characteristic function \( f \) has degree at most 2, and suppose that \( C(f) \leq n - 2 \). Let \( C_1(f) \) the maximum certificate complexity of any \( m \in F \).
The family \( F \) is contained in the union of at most \( T \) many \( C_1(f) \)-cosets, where
\[
T = \frac{2[C_1(f)/2](C_1(f) - 1)!}{2^{[C_1(f)/2]}}.
\]

Moreover, \( C_1(f) \geq 2 \).

Proof. Let \( \mu \in F \) be a perfect matching satisfying \( C(f, \mu) = C_1(f) \). Lemma 2.11 applied to \( m_1 = m_2 = \mu \), shows that \( C(f, \mu) \geq 2 \), and so \( C_1(f) \geq 2 \). Lemma 4.2 shows that \( F \) is \( r \)-covered, where \( r = C(f, \mu) - 1 \). Note that \( r \leq C(f) - 1 \leq n - 3 \). Without loss of generality, we can assume that \( F \subseteq [1 \leftrightarrow r + 1] \cup [2 \leftrightarrow r + 2] \cup \cdots \cup [r \leftrightarrow 2r] \).

Lemma 4.3 shows that every \( m \in F \) has a certificate of the form \( \{\{1, m(1)\}, \ldots, \{r, m(r)\}, \{i, j\}\} \), which we call the standard certificate. The lemma also implies that \( F \) is not \((r - 1)\)-covered: otherwise, every perfect matching in \( F \) would have a certificate of size \( r < C_1(f) \), which contradicts the definition of \( C_1(f) \).

A standard certificate \( \{\{1, m(1)\}, \ldots, \{r, m(r)\}, \{i, j\}\} \) satisfies \( m(1), \ldots, m(r) \notin [r] \), since otherwise Lemma 4.3 would imply that \( F \) is \((r - 1)\)-covered (since some of the pairs \( \{t, m(t)\} \) coincide), which is impossible. This allows us to color the pairs in a standard certificate unambiguously: a pair \( \{t, m(t)\} \) is colored \( t \), and a pair \( \{i, j\} \) with \( i, j \notin [r] \) is colored \( r + 1 \). Every standard certificate contains exactly one pair of each color.

The remainder of the proof is identical to that of Lemma 3.3.

We can now prove Theorem 1.4. The proof is very similar to that of Theorem 1.3.

Proof of Theorem 1.4. Let \( F \) be a 2-intersecting subset of \( \mathcal{M}_{2n} \) of size \((2n - 5)!\), where \( n \geq 2 \), and let \( f \) be its characteristic function. Our goal is to show that
\[
F = \{m \in \mathcal{M}_{2n} : m(i_1) = j_1 \text{ and } m(i_2) = j_2\}
\]

for some distinct \( i_1, j_1, i_2, j_2 \in [2n] \).

Suppose first that \( n \geq 8 \). Since \( n \geq 3 \), Theorem 2.8 shows that \( F \) is maximal and that \( \deg f \leq 2 \). Lemma 4.12 shows that \( C_1(f) \leq n - 2 \). Lemma 4.4 implies that \( C_1(f) \geq 2 \) and \( |F| \leq T(2n - 2C_1(f) - 1)! / 2^{[C_1(f)/2]} \). Therefore
\[
\frac{(2n - 5)!!}{(2n - 2C_1(f) - 1)!!} \leq \frac{2[C_1(f)/2](C_1(f) - 1)!}{2^{[C_1(f)/2]}}.
\]
The left-hand side is clearly increasing in \( n \). Since \( n \geq C_1(f) + 2 \), it follows that
\[
\frac{(2C_1(f))!}{3 \cdot 2^{C_1(f)}} = \frac{(2C_1(f) - 1)!!}{3} \leq \frac{2[C_1(f)/2](C_1(f) - 1)!}{2^{[C_1(f)/2]}},
\]
and so
\[
(2C_1(f))! \leq 3 \cdot 2^{[C_1(f)/2]}(2C_1(f)/2)!(C_1(f) - 1)!.
\]

If \( C_1(f) \) is even then this reads
\[
(2C_1(f))! \leq 3 \cdot 2^{C_1(f)/2}C_1(f)^2.
\]
When \( C = C_1(f) \) increases by 1, the left-hand side increases by a factor of \((2C + 2)(2C + 1)\), while the right-hand side increases by a factor of \( \sqrt{2}(C + 1)^2 \), which is smaller for \( C \geq 2 \). When \( C = 3 \), the inequality fails. Therefore in this case, \( C_1(f) = 2 \).

If \( C_1(f) \) is odd then the inequality reads
\[
(2C_1(f))! \leq 3 \cdot 2^{(C_1(f)+1)/2}C_1(f)! \cdot (C_1(f) - 1)! < 3 \cdot 2^{(C_1(f)+1)/2}C_1(f)^2.
\]
As before, if we increase \( C = C_1(f) \) by 1, the left-hand side increases by a larger factor than the (far) right-hand side. When \( C = 3 \), the inequality fails, and so this case cannot happen.
Now let \( m \in \mathcal{F} \) be arbitrary. Since \( C_1(f) = 2 \), there is a certificate of the form \( \{i_1, j_1\}, \{i_2, j_2\} \) for \( m \). The number of permutations satisfying this certificate is \( (2n - 5)!! \), and we conclude that \( \mathcal{F} \) consists of all perfect matchings satisfying the certificate, completing the proof in this case.

In order to complete the proof, we consider the case \( n \leq 7 \). It suffices to show that for \( n \in \{2, 3, 4, 5, 6, 7\} \), any 2-intersecting family of size \( (2n - 5)!! \) which contains the perfect matching \( \{1, n + 1\}, \ldots, \{n + 2n\} \) is of the form

\[
\{ m \in \mathcal{M}_{2n} : m(i_1) = n + i_1, m(i_2) = n + i_2 \}
\]

for some \( i_1 \neq i_2 \in [n] \).

If \( n = 2 \) or \( n = 3 \) then \( (2n - 5)!! = 1 \), and the claim can be verified directly. In order to verify the remaining cases \( n \in \{4, 5, 6, 7\} \), we use exhaustive search just as in the proof of Theorem 1.3. The relevant code appears in Appendix A.3.

References


18


A SAGE code

The code snippets below form part of the proofs of Lemma 2.5 and Theorems 1.3 and 1.4. They are also available as part of the arXiv version of the paper.
A.1 Code for Lemma 2.5

```python
import itertools

def degree_2_functions():
    "Generate vector space of degree 2 functions, represented as truth table"
    return span(QQ, [vector(int((x & m) == m) for x in range(16)) for m in [0, 1, 2, 4, 8, 3, 5, 9, 6, 10, 12]])

def verify_no_sensitivity_4():
    "Verify that no function from \{0,1\}^4 to \{0,1\} has sensitivity 4 at zero"
    degree_2 = degree_2_functions()
```

A.2 Code for Theorem 1.3

```python
def intersection_size(a, b):
    "Size of intersection of two permutations"
    return sum(i==j for (i,j) in zip(a,b))

def generate_G_n(n, t):
    "Generate graph whose cliques are t-intersecting families of S_n containing id"
    idp = list(range(n))
    V = [a for a in Permutations(range(n)) if intersection_size(a, idp) >= t]
    E = [(a,b) for a in V for b in V if t <= intersection_size(a, b) < n]
    return Graph([V, E])

def verify_main_theorem_n(n):
    "Verify that all 2-intersecting families containing id are 2-cosets for given n"
    G_n = generate_G_n(n, 2)
    return G_n.clique_number() == factorial(n - 2) and len(G_n.clique_maximum()) == binomial(n, 2)

def verify_main_theorem():
    "Verify that all 2-intersecting families containing id are 2-cosets for 4 <= n <= 7"
    return all(verify_main_theorem_n(n) for n in [4,5,6,7])
```

A.3 Code for Theorem 1.4

```python
def intersection_size(a, b):
    "Size of intersection of two perfect matchings"
    return len(a.intersection(b))

def generate_G_n(n, t):
    "Generate graph whose cliques are t-intersecting families of M_2n containing a fixed PM"
    idm = frozenset(PerfectMatchings(2*n).__iter__().__next__())
    V = [frozenset(a) for a in PerfectMatchings(2*n) if intersection_size(frozenset(a), idm) >= t]
    E = [(a,b) for a in V for b in V if t <= intersection_size(a, b) < n]
    return Graph([V, E])

def verify_main_theorem_n(n):
    "Verify that all 2-intersecting families containing a fixed PM are 2-cosets for given n"
    G_n = generate_G_n(n, 2)
    return G_n.clique_number() == factorial(2*n-4)/(2**(n-2)*factorial(n-2)) and len(G_n.clique_maximum()) == binomial(n, 2)

def verify_main_theorem():
    "Verify that all 2-intersecting families containing a fixed PM are 2-cosets for 4 <= n <= 7"
    return all(verify_main_theorem_n(n) for n in [4,5,6,7])
```