Skip Chains

Yuval Filmus

November 2010

1 Introduction

In this note, we study self-avoiding walks on the integer numbers starting at the origin and at each step moving to a point a distance of at most two away.

More formally, a *skip chain* is a finite or infinite sequence $p_0, p_1, \ldots$ of integers satisfying the following properties:

1. Origin: $p_0 = 0$.
2. Moves: $|p_{i+1} - p_i| \leq 2$.
3. Self-avoidance: $p_i \neq p_j$ for $i \neq j$.

We will be particularly interested in the quantity

$$\lim_{n \to \infty} w_n^{1/n},$$

were w_n is the number of walks of length n.

2 Reduction to automaton

Consider a skip chain p_i. The skip chain is completely described by the sequence of moves $p_{i+1} - p_i \in \{\pm 1, \pm 2\}$. The moves that the skip chain can make at a given point depend on the location of the already visited squares relative to its current position. These can be described as doubly infinite words over the alphabet $\{\boxminus, \boxplus, \boxtimes\}$, whose meaning is never visited, visited, current, respectively. Some more notation: $\boxdot$ will denote either $\boxminus$ or $\boxplus$; w^n will denote the word w repeated n times; $\boxminus^{\leftarrow}, \boxminus^{\rightarrow}$ will denote a left-infinite (right-infinite) word composed of $\boxminus$; $\boxdot^{\leftarrow}, \boxminus^{\rightarrow}$ will denote a left-infinite (right-infinite) word composed of $\boxminus, \boxplus$ in arbitrary position.

We define the following eight families of states:

$$\begin{aligned}
A_n^R &= \boxminus^{\leftarrow} (\boxplus\boxminus)^n \boxtimes \boxminus^{\rightarrow}, && n \geq 0,\\
B_n^R &= \boxminus^{\leftarrow} (\boxplus\boxminus)^n \boxplus \boxtimes\boxminus^{\rightarrow} && n \geq 0,\\
C_n^R &= \boxminus^{\leftarrow} (\boxminus\boxplus)^n \boxtimes \boxplus\boxminus^{\rightarrow} && n \geq 1,\\
D_n^R &= \boxdot^{\leftarrow} \boxplus (\boxplus\boxminus)^n \boxtimes \boxminus^{\rightarrow} && n \geq 1,\\
E_n^R &= \boxdot^{\leftarrow} \boxplus (\boxplus\boxminus)^n \boxplus \boxtimes\boxminus^{\rightarrow} && n \geq 0,\\
F_n^R &= \boxminus^{\leftarrow} (\boxminus\boxplus)^n \boxtimes \boxplus \boxplus \boxdot^{\rightarrow} && n \geq 0,\\
G_n^R &= \boxdot^{\leftarrow} (\boxplus\boxminus)^n \boxplus \boxtimes \boxplus \boxminus^{\rightarrow} && n \geq 1,\\
H_n^R &= \boxdot^{\leftarrow} (\boxplus\boxminus)^n \boxplus \boxtimes \boxplus \boxplus\boxdot^{\rightarrow} && n \geq 0.
\end{aligned}$$

Eight more families are obtained by reversing the words. These will be denoted by A_n^L etc. Note that $A_0^R = A_0^L$, and otherwise all states are different. We denote $A_0 = A_0^R = A_0^L$.

Using these states, we can describe an infinite deterministic automaton which *accepts* a sequence of moves iff it leads to a skip chain. The states of the automaton are the infinite families of states mentioned above, and an absorbing error state X. A sequence of moves is a skip chain iff it doesn't end up at X.

We now describe the automaton. The starting state is A_0. The following table describes the allowable moves for the *right* families of states (*left* families are obtained by reversing all directions). When two target states are given, the first corresponds to the member of the family with smallest n, in other words $A_0, B_0^R, C_1^R, D_1^R, E_0^R, F_0^R, G_1^R, H_0^R$, and the second corresponds to all other members.

	-2	-1	$+1$	$+2$
A_n^R	A_1^L, X	B_0^L, C_n^R	B_n^R	A_{n+1}^R
B_n^R	E_0^L, F_{n-1}^R	X	E_0^R	D_1^R
C_n^R	E_0^L, F_{n-2}^R	X	X	E_0^R
D_n^R	X	F_0^L, G_{n-1}^R	E_n^R	D_{n+1}^R
E_n^R	X, H_{n-1}^R	X	E_0^R	D_1^R
F_n^R	E_0^L, F_{n-1}^R	X	X	X
G_n^R	H_{n-1}^R	X	X	E_0^R
H_n^R	X, H_{n-1}^R	X	X	X

3 Infinite skip chains

Using the automaton described above, we can explicitly list all maximal skip chains. A maximal skip chain is either a finite skip chain which cannot be extended, or an infinite skip chain. Every finite skip chain can be extended to some maximal skip chain, so the set of all finite skips is equal to the set of all prefixes of all maximal skip chains.

At each point after the first move, the automaton is either in a right state or a left state. When in a right state, the notation $\mathrm{B}_2, \mathrm{B}_1, \mathrm{F}_1, \mathrm{F}_2$ will represent the moves $-2, -1, +1, +2$; when in a left state, these will represent the opposite moves $+2, +1, -1, -2$. In the starting position, both directions are the same, and we will use $\mathrm{F}_1, \mathrm{F}_2$ to mean $\pm 1, \pm 2$; the direction of the first move determines the direction of the second state.

Tedious but elementary calculations yield the following recursive description of all maximal skip chains. For $n \geq 0$, denote by $\mathcal{E}_n$ the set of all maximal skip chains starting at state E_n^R/E_n^L, and denote by $\mathcal{A}_0$ the set of all chains. These are defined by the following equations:

$$\begin{aligned}
\mathcal{A}_0 &= \{\mathrm{F}_2^n \mathrm{F}_1 \mathrm{F}_2 \mathrm{B}_1 \mathrm{B}_2 \mathcal{E}_0 : n \geq 0\} \cup \{\mathrm{F}_2^n \mathrm{F}_1 \mathrm{F}_2^m \mathrm{F}_1 \mathcal{E}_m : n \geq 0,\, m \geq 1\} \\
&\quad \cup \{\mathrm{F}_2^n \mathrm{F}_1 \mathrm{F}_2^{m+1} \mathrm{B}_1 \mathrm{B}_2^m : n \geq 0\, m \geq 1\} \cup \{\mathrm{F}_2^n \mathrm{F}_1 \mathrm{F}_2^m \mathrm{B}_1 \mathrm{F}_2 \mathcal{E}_0 : n \geq 0\, m \geq 2\} \\
&\quad \cup \{\mathrm{F}_2^n \mathrm{F}_1^2 \mathcal{E}_0 : n \geq 0\} \cup \{\mathrm{F}_2^n \mathrm{F}_1 \mathrm{B}_2^{n+1} \mathcal{E}_0 : n \geq 0\} \\
&\quad \cup \{\mathrm{F}_2^n \mathrm{B}_1 \mathrm{F}_2 \mathcal{E}_0 : n \geq 1\} \cup \{\mathrm{F}_2^n \mathrm{B}_1 \mathrm{B}_2^n \mathcal{E}_0 : n \geq 1\}, \\
\mathcal{E}_n &= \mathrm{B}_2^n \cup \mathrm{F}_1 \mathcal{E}_0 \cup \mathrm{F}_2 \mathrm{B}_1 \mathrm{B}_2 \mathcal{E}_0 \\
&\quad \cup \{\mathrm{F}_2^m \mathrm{F}_1 \mathcal{E}_m : m \geq 1\} \cup \{\mathrm{F}_2^{m+1} \mathrm{B}_1 \mathrm{B}_2^m : m \geq 1\} \cup \{\mathrm{F}_2^m \mathrm{B}_1 \mathrm{F}_2 \mathcal{E}_0 : m \geq 2\}.
\end{aligned}$$

4 Asymptotics

Denote by $\mathcal{A}_0^{[\ell]}, \mathcal{E}_n^{[\ell]}$ the number of prefixes of the given set of length ℓ. Using the Iverson bracket notation, we can calculate explictly

$$\begin{aligned}
\mathcal{E}_n^{[\ell]} &= [n \leq \ell] + [\ell \geq 2] + [\ell \geq 3] \\
&\quad + \mathcal{E}_0^{[\ell-1]} + \mathcal{E}_0^{[\ell-3]} + \mathcal{E}_1^{[\ell-2]} \\
&\quad + \sum_{m=2}^{\ell-2} \left([\ell \leq 2m] + \mathcal{E}_m^{[\ell-m-1]} + \mathcal{E}_0^{[\ell-m-2]} \right).
\end{aligned}$$

Therefore

$$\mathcal{E}_n^{[\ell]} = \mathcal{E}_0^{[\ell-1]} + \mathcal{E}_0^{[\ell-3]} + \mathcal{E}_1^{[\ell-2]} + \sum_{m=2}^{\ell-2} \left(\mathcal{E}_m^{[\ell-m-1]} + \mathcal{E}_0^{[\ell-m-2]} \right) + \epsilon, \quad 0 \leq \epsilon \leq \ell.$$

In order to estimate $\mathcal{E}_n^{[\ell]}$, we define recurrence equations which will provide both a lower bound and an upper bound on $\mathcal{E}_n^{[\ell]}$, independent of ℓ. These are

$$\begin{aligned}
L_\ell &= L_{\ell-1} + L_{\ell-2} + L_{\ell-3} + \sum_{m=2}^{\ell-2} (L_{\ell-m-1} + L_{\ell-m-2}), & L_0 &= 1, \\
U_\ell &= U_{\ell-1} + U_{\ell-2} + U_{\ell-3} + \sum_{m=2}^{\ell-2} (U_{\ell-m-1} + U_{\ell-m-2}) + \ell, & U_0 &= 1.
\end{aligned}$$

Consider the sequence L_ℓ. When expanding out all terms, each sequence leading to $L_0 = 1$ corresponds to a selection of ℓ_i such that $\sum_i \ell_i = \ell$. The choices for ℓ_i are the multiset

$$\{1, 2, 3\} \cup \{m, m+1 : m \geq 3\} = \{1, 2\} \cup \{m, m : m \geq 3\}.$$

Therefore L_ℓ is the coefficient of x_ℓ in the generating series

$$\sum_{t=0}^{\infty}\left(x+x^2+2\frac{x^3}{1-x}\right)=\frac{1}{1-x-x^2-2x^3/(1-x)}=\frac{1-x}{1-2x-x^3}.$$

The denominator has one real root $\mu^{-1}\approx 0.453397651516404$ which is also the root of smallest modulus, and so $L_\ell=\Theta(\mu^\ell)$, where $\mu\approx 2.20556943040059$.

In order to deal with the upper bound sequence, fix some term U_k that we wish to estimate. We then form a new sequence V_ℓ by replacing the additive constant ℓ in the recurrence relation for V_ℓ with V_0. Clearly, for $\ell\le k$ we have $V_\ell\le kV_\ell$. When expanding out all terms corresponding to V_ℓ, each sequence leading to V_0 corresponds to a selection of ℓ_i such that $\sum_i \ell_i\le\ell$, and the ℓ_i are chosen as in the sequence L_ℓ. Therefore V_ℓ is the coefficient of x^ℓ in the generating series

$$\frac{1-x}{1-2x-x^3}\cdot\frac{1}{1-x}=\frac{1}{1-2x-x^3},$$

of the same order of growth as L_ℓ. In particular, we get $U_\ell=\mathrm{O}(\ell\mu^\ell)$. Since $L_\ell\le\mathcal{E}_n^{[\ell]}\le U_\ell$, we conclude that for all n uniformly,

$$\Omega(\mu^\ell)\le\mathcal{E}_n^{[\ell]}\le\mathrm{O}(\ell\mu^\ell).$$

We now go back to $\mathcal{A}_0^{[\ell]}$. Since $\mathrm{F}_1^2\mathcal{E}_0^{[\ell-2]}\subset\mathcal{A}_0^{[\ell]}$, we see that $\mathcal{A}_0^{[\ell]}$ grows at least as fast as $\mathcal{E}_0^{[\ell]}$ (up to constants). On the other hand, each sequence in $\mathcal{A}_0^{[\ell]}$ is of the form $w\mathcal{E}_m^{\ell-k}$ for some m,k (for the finite sequences, take $k=\ell$). Inspection of the equation for $\mathcal{A}_0$ reveals that there are $O(\ell^2)$ choices for w. Therefore

$$\Omega(\mu^\ell)\le\mathcal{A}_0^{[\ell]}\le\mathrm{O}(\ell^3\mu^\ell).$$

We conclude that

$$\lim_{\ell\to\infty}\sqrt[\ell]{\mathcal{A}_0^{[\ell]}}=\mu.$$