One-step modified log Sobolev on the complete complex

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Suppose that ν_k is the uniform distribution over $\binom{[n]}{k}$, and let μ_k be an arbitrary distribution over $\binom{[n]}{k}$.

Define ν_{k-1}, μ_{k-1} to be the distributions obtained by removing a random element from a set sampled according to ν_k, μ_k (respectively). Note that ν_{k-1} is just the uniform distribution over $\binom{[n]}{k-1}$.

We will show:

$$\frac{1}{k}\mathrm{D}(\mu_k \| \nu_k) \geq \frac{1}{k-1}\mathrm{D}(\mu_{k-1} \| \nu_{k-1}). \tag{MLS}$$

Since ν_k is the uniform distribution, we can compute

$$\mathrm{D}(\mu_k \| \nu_k) = \sum_x \mu_k(x) \log \frac{\mu_k(x)}{\nu_k(x)} = \log \binom{n}{k} - H(\mu_k) = H(\nu_k) - H(\mu_k),$$

and similarly

$$\mathrm{D}(\mu_{k-1} \| \nu_{k-1}) = H(\nu_{k-1}) - H(\mu_{k-1}).$$

Hence (MLS) is equivalent to

$$\frac{H(\mu_{k-1})}{k-1} - \frac{H(\mu_k)}{k} \geq \frac{H(\nu_{k-1})}{k-1} - \frac{H(\nu_k)}{k}.$$

In other words, we need to show that the left-hand side is minimized when μ_k is uniform.

Let $X_1, \ldots, X_k$ be the elements of a random set sampled according to μ_k, in random order. Then

$$H(\mu_k) = H(X_1, \ldots, X_{k-1}) + H(X_k | X_1, \ldots, X_{k-1}) = H(\mu_{k-1}) + H(X_k | X_1, \ldots, X_{k-1}),$$

and so

$$kH(\mu_{k-1}) - (k-1)H(\mu_k) = H(\mu_{k-1}) - (k-1)H(X_k | X_1, \ldots, X_{k-1}).$$

Similarly,

$$\begin{aligned} H(\mu_{k-1}) &= H(X_1) + H(X_2|X_1) + \cdots + H(X_{k-1}|X_1, \ldots, X_{k-2}) \\ &= H(X_k) + H(X_k|X_1) + \cdots + H(X_k|X_1, \ldots, X_{k-2}), \end{aligned}$$

due to symmetry. Therefore

$$\begin{aligned} kH(\mu_{k-1}) - (k-1)H(\mu_k) &= \sum_{i=1}^{k-1} [H(X_k|X_1, \ldots, X_{i-1}) - H(X_k|X_1, \ldots, X_{k-1})] \\ &= \sum_{i=1}^{k-1} I(X_k; X_i, \ldots, X_{k-1} | X_1, \ldots, X_{i-1}). \end{aligned}$$

Given $X_1, \ldots, X_{i-1}$, the values $(X_i, \ldots, X_k)$ are (up to renaming) some distribution over $\binom{[n-(i-1)]}{k-(i-1)}$, and so to complete the proof, it suffices to show that $I(X_k; X_1, \ldots, X_{k-1})$ is minimized for the uniform distribution over $\binom{[n]}{k}$.

Indeed, convexity of mutual information shows that given the distribution of $(X_1, \ldots, X_{k-1})$, the mutual information $I(X_k; X_1, \ldots, X_{k-1})$ is minimized when X_k is chosen uniformly over $[n] \setminus \{X_1, \ldots, X_{k-1}\}$. Applying this repeatedly for all variables, we obtain that $I(X_k; X_1, \ldots, X_{k-1})$ is minimized when $(X_1, \ldots, X_k)$ is the uniform distribution.