Kahn–Kalai conjecture: an exposition after Jinyoung Park and Huy Tuan Pham

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Abstract

Jinyoung Park and Huy Tuan Pham recently proved the Kahn–Kalai conjecture (arXiv:2203.17207). We slightly rephrase their proof.

1 Introduction

Let $f\colon \{0,1\}^n \to \{0,1\}$ be a non-constant monotone Boolean function. We identify inputs of f with subsets of $[n] = \{1,\ldots,n\}$. For $p \in [0,1]$ we define $\mu_p(f) = \mathbb{E}_{x\sim\mu_p}[f(x)]$, where μ_p is the product distribution on $\{0,1\}^n$ whose marginals satisfy $\Pr[x_i = 1] = p$. Since f is not constant, $\mu_p(f)$ is strictly increasing in p, and so there is a unique value p_c such that $\mu_{p_c}(f) = 1/2$. We call this the *critical probability* of f, denoted $p_c(f)$.

A *minterm* of f is a set S such that $f(S) = 1$ but $f(S') = 0$ for all $S' \subsetneq S$. We denote the set of minterms of f by $\mathsf{minterms}(f)$. The expected number of minterms of f which evaluate to 1 under a random input distributed according to μ_q is

$$\mathsf{mcost}_q(f) = \sum_{S\in\mathsf{minterms}(f)} q^{|S|}.$$

The union bound shows that $\mu_q(f) \le \mathsf{mcost}_q(f)$, and consequently, if $\mathsf{mcost}_q(f) \le 1/2$ then $p_c(f) \ge q$.

We can sometimes obtain a better lower bound on $p_c(f)$ by considering weakenings of f, which are non-constant monotone Boolean functions $g\colon \{0,1\}^n \to \{0,1\}$ satisfying $g(x) \ge f(x)$ for all $x \in \{0,1\}^n$, a condition which we succinctly denote by $g \ge f$. If $g \ge f$ then $\mu_q(f) \le \mu_q(g) \le \mathsf{mcost}_q(g)$. This prompts the following definition:

$$\mathsf{cost}_q(f) = \min_{g\ge f} \mathsf{mcost}_q(g),$$

where the minimum is over all monotone Boolean functions g. If $\mathsf{cost}_q(f) \le 1/2$ then $\mu_q(f) \le 1/2$, and so $p_c(f) \ge q$. If f is non-constant then the function $\mathsf{cost}_q(f)$ is strictly increasing in q, and so there is a unique value q such that $\mathsf{cost}_q(f) = 1/2$. We call this the *expectation threshold* of f, denoted $q(f)$.

We have seen that $p_c(f) \ge q(f)$. Kahn and Kalai [KK07] conjectured that this bound is almost tight:

$$p_c(f) = O(q(f)\log \ell(f)),$$

where $\ell(f)$ is the size of the largest minterm of f. In this short note, we give an exposition of the proof of the Kahn–Kalai conjecture due to Park and Pham [PP22].

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2 Main lemma

The main idea of the proof is encapsulated in the lemma which we prove in this section. In order to state the lemma, we need one more piece of notation. If $f\colon \{0,1\}^n \to \{0,1\}$ and $W \subseteq [n]$, we let $f|_{W\leftarrow 1}\colon \{0,1\}^{[n]\setminus W} \to \{0,1\}$ be the function obtained by fixing the coordinates in W to 1, that is, $f|_{W\leftarrow 1}(S) = f(S \cup W)$.

Lemma 1. *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a monotone Boolean function.*

For a set $W \subseteq [n]$ and a threshold r, we define two functions g, h as follows:

- *g is the monotone Boolean function whose minterms are $\{S \in \mathsf{minterms}(f|_{W\leftarrow 1}) : |S| \geq r\}$.*
- *h is the monotone Boolean function whose minterms are $\{S \in \mathsf{minterms}(f|_{W\leftarrow 1}) : |S| < r\}$.*

Then $\mathsf{cost}_q(f) \leq \mathsf{mcost}_q(g) + \mathsf{cost}_q(h)$ for all $0 \leq q \leq 1$, and $\ell(h) < r$.

Furthermore, the following holds for all $C \geq 1$ and all $0 \leq q \leq 1/C$:

$$\mathop{\mathbb{E}}_{W\sim\mu_{Cq}}[\mathsf{mcost}_q(g)] \leq \frac{2^{\ell(f)}}{C^r}.$$

Proof. Clearly $\ell(h) < r$. Let us explain why $\mathsf{cost}_q(f) \leq \mathsf{mcost}_q(g) + \mathsf{cost}_q(h)$. Suppose that $\mathsf{cost}_q(h) = \mathsf{mcost}_q(k)$, where $k \geq h$. We have $f|_{W\leftarrow 1} = \max(g,h) \leq \max(g,k)$. Since $\mathsf{minterms}(\max(g,k)) \subseteq \mathsf{minterms}(g) \cup \mathsf{minterms}(k)$, it follows that $\mathsf{cost}_q(f|_{W\leftarrow 1}) \leq \mathsf{mcost}_q(g) + \mathsf{mcost}_q(k) = \mathsf{mcost}_q(g) + \mathsf{cost}_q(h)$. If we treat $f|_{W\leftarrow 1}$ as a function on $\{0,1\}^n$ which doesn't depend on the coordinates in W then $f \leq f|_{W\leftarrow 1}$, and so $\mathsf{cost}_q(f) \leq \mathsf{mcost}_q(g) + \mathsf{cost}_q(h)$.

The main content of the lemma is the bound on $\mathbb{E}[\mathsf{mcost}_q(g)]$. The starting point is the formula

$$\mathop{\mathbb{E}}_{W\sim\mu_{Cq}}[\mathsf{mcost}_q(g)] = \mathop{\mathbb{E}}_{W\sim\mu_{Cq}} \sum_{\substack{T\in\mathsf{minterms}(f|_{W\leftarrow 1})\\ |T|\geq r}} q^{|T|}.$$

We now perform a change of variables. Let $Z = W \cup T$, so that $W = Z \setminus T$ (since $T \subseteq [n] \setminus W$). Then

$$\mathop{\mathbb{E}}_{W\sim\mu_{Cq}}[\mathsf{mcost}_q(g)] = \mathop{\mathbb{E}}_{Z\sim\mu_{Cq}} \sum_{\substack{T\subseteq Z\\ T\in\mathsf{minterms}(f|_{Z\setminus T\leftarrow 1})\\ |T|\geq r}} \frac{\mu_{Cq}(W)}{\mu_{Cq}(Z)} q^{|T|}.$$

Here $\mu_{Cq}(W) = \Pr_{x\sim\mu_{Cq}}[x = W] = (Cq)^{|W|}(1-Cq)^{n-|W|}$. Since $|Z| = |W| + |T|$ and $|T| \geq r$,

$$\frac{\mu_{Cq}(W)}{\mu_{Cq}(Z)} q^{|T|} = \left(\frac{1-Cq}{Cq}\right)^{|T|} q^{|T|} \leq \frac{1}{C^{|T|}} \leq \frac{1}{C^r}.$$

It follows that

$$\mathop{\mathbb{E}}_{W\sim\mu_{Cq}}[\mathsf{mcost}_q(g)] \leq \frac{1}{C^r} \mathop{\mathbb{E}}_{Z\sim\mu_{Cq}} \#\{T \subseteq Z : T \in \mathsf{minterms}(f|_{Z\setminus T\leftarrow 1})\}.$$

In order to complete the proof, we bound the size of the set on the right-hand side by $2^{\ell(f)}$.

Given Z, suppose that there exists $T \subseteq Z$ such that $T \in \mathsf{minterms}(f|_{Z\setminus T\leftarrow 1})$. In particular, $f|_{Z\setminus T\leftarrow 1}(T) = 1$, and so $f(Z) = 1$. Consequently, Z is a superset of some minterm of f. Fix an arbitrary such minterm S_Z.

Given Z and a subset $S_Z \subseteq Z$ which is a minterm of f, we now show that if $T \subseteq Z$ satisfies $T \in \mathsf{minterms}(f|_{Z\setminus T\leftarrow 1})$, then $T \subseteq S_Z$. Since $|S_Z| \leq \ell(f)$, this would show that there are at most $2^{\ell(f)}$ many sets $T \subseteq Z$ such that $T \in \mathsf{minterms}(f|_{Z\setminus T\leftarrow 1})$.

Suppose that $T \in \mathsf{minterms}(f|_{Z\setminus T\leftarrow 1})$, and let $i \in T$. Since T is a minterm of $f|_{Z\setminus T\leftarrow 1}$, in particular $f|_{Z\setminus T\leftarrow 1}(T \setminus \{i\}) = 0$, and so $f(Z \setminus \{i\}) = 0$. Since S_Z is a minterm of f, this means that $i \in S_Z$ (otherwise $Z \setminus \{i\} \supseteq S_Z$). As i was arbitrary, we conclude that $T \subseteq S_Z$, completing the proof. $\square$

We can think of the operation $f \mapsto f|_{W\leftarrow 1}$ as a kind of random restriction. The reweighting scheme used in the proof is reminiscent of arguments in Håstad's proof of shrinkage [Hås98], and in Razborov's proof of the switching lemma [Raz95, Bea94].

3 Main theorem

Let $f\colon \{0,1\}^n \to \{0,1\}$ be a non-constant monotone Boolean function. Let $C \geq 1$ and $q \leq 1/C$ be parameters to be chosen later. We define a random sequence of functions $f_0, \ldots, f_T$, where $f_i\colon \{0,1\}^{D_i} \to \{0,1\}$, as follows.

The starting point is $f_0 = f$ (and so $D_0 = [n]$).

Given f_{i-1}, we choose a subset $W_i \subseteq D_{i-1}$ according to μ_{Cq}, let $D_i = D_{i-1} \setminus W_i$, and define two functions $g_i, f_i\colon \{0,1\}^{D_i} \to \{0,1\}$ as follows:

- g_i is the monotone Boolean function whose minterms are $\{S \in \mathsf{minterms}(f_{i-1}|_{W_i \leftarrow 1}) : |S| \geq \ell(f_{i-1})/2\}$.
- f_i is the monotone Boolean function whose minterms are $\{S \in \mathsf{minterms}(f_{i-1}|_{W_i \leftarrow 1}) : |S| < \ell(f_{i-1})/2\}$.

(Recall that $\ell(f)$ is the maximum size of a minterm of f.)

The sequence terminates once we reach a constant function f_T. By construction, $\ell(f_T) < \ell(f)/2^T$ for $T \geq 1$, and so $T \leq \max(\log_2 \ell(f), 1) \leq \log_2 \ell(f) + 1$.

Finally, we extend the definition of D_i, W_i all the way to $T_{\max} = \log_2 \ell(f) + 1$, and let

$$W = W_1 \cup \cdots \cup W_{T_{\max}}.$$

Note that $W \sim \mu_p$, where

$$1 - p = (1 - Cq)^{T_{\max}}.$$

If S is a minterm of f_i then it is a minterm of $f_{i-1}|_{W_i \leftarrow 1}$, and so $f_{i-1}(S \cup W_i) = 1$. Consequently, if S is a minterm of f_i then $f(S \cup W_1 \cup \cdots \cup W_i) = 1$. In particular, if $\emptyset$ is a minterm of f_T, equivalently if $f_T \equiv 1$, then $f(W_1 \cup \cdots \cup W_T) = 1$, and so $f(W) = 1$.

Lemma 1 shows that for each choice of $W_1, \ldots, W_{T_{\max}}$,

$$\mathsf{cost}_q(f) \leq \sum_{i=1}^{T} \mathsf{mcost}_q(g_i) + \mathsf{cost}_q(f_T).$$

If $f_T \equiv 0$ then $\mathsf{cost}_q(f_T) \leq \mathsf{mcost}_q(f_T) = 0$, and if $f_T \equiv 1$ then $\mathsf{cost}_q(f_T) = \mathsf{mcost}_q(f_T) = 1$. As noted above, if $f_T \equiv 1$ then $f(W) = 1$, and so $\mathsf{cost}_q(f) \leq f(W)$.

Applying Lemma 1 to bound $\mathbb{E}[\mathsf{mcost}_q(g_i)]$, we see that

$$\mathsf{cost}_q(f) \leq \mathop{\mathbb{E}}_{W_1,\ldots,W_{T_{\max}}} \left[\sum_{i=1}^{T} \mathsf{mcost}_q(g_i) + \mathsf{cost}_q(f_T) \right] \leq \sum_{i=1}^{T} \frac{2^{\ell(f_{i-1})}}{C^{\ell(f_{i-1})/2}} + \mu_p(f).$$

We now choose $C = 64$, so that

$$\mathsf{cost}_q(f) \leq \sum_{i=1}^{T} \frac{1}{4^{\ell(f_{i-1})}} + \mu_p(f).$$

By construction, the sequence $\ell(f_0), \ldots, \ell(f_{T-1})$ is strictly decreasing, and $\ell(f_{T-1}) \geq 1$. Therefore

$$\sum_{i=1}^{T} \frac{1}{4^{\ell(f_{i-1})}} \leq \sum_{\ell=1}^{\infty} \frac{1}{4^{\ell}} = \frac{1}{3},$$

and so

$$\mathsf{cost}_q(f) \leq \frac{1}{3} + \mu_p(f).$$

We now choose q:

$$q = \frac{1 - (1 - p_c(f))^{1/4T_{\max}}}{C}.$$

Equivalently, we choose q by solving the equation

$$1 - p_c(f) = (1 - Cq)^{4T_{\max}}.$$

By construction,

$$(1-p)^4 = \left((1-Cq)^{T_{\max}}\right)^4 = 1 - p_c(f).$$

Therefore, if we sample $S_1, \ldots, S_4 \sim \mu_p$ then $S_1 \cup \cdots \cup S_4 \sim \mu_{p_c(f)}$, and so

$$\frac{1}{2} = \Pr_{S \sim \mu_{p_c(f)}}[f(S) = 0] \leq \Pr_{S_1,\ldots,S_4 \sim \mu_p}[f(S_1) = \cdots = f(S_4) = 0] = (1 - \mu_p(f))^4,$$

implying that

$$\mu_p(f) \leq 1 - \left(\frac{1}{2}\right)^{1/4} \leq \frac{1}{6},$$

and so

$$\mathsf{cost}_q(f) \leq \frac{1}{2}.$$

Consequently, $q(f) \geq q$.

It remains to understand the relation between $p_c(f)$ and q. We have

$$1 - p_c(f) = (1 - 64q)^{4T_{\max}} \geq 1 - 256T_{\max}q,$$

and so

$$p_c(f) \leq 256T_{\max}q \leq 256(\log_2 \ell(f) + 1)q(f).$$

Summarizing, we have proved the Kahn–Kalai conjecture:

Theorem 2. *If $f \colon \{0,1\}^n \to \{0,1\}$ is a non-constant monotone Boolean function then*

$$p_c(f) = O(q(f)(\log \ell(f) + 1)).$$

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