An Equational Proof

Let's prove that if $(x, y) = 1$ then $(x + y, xy) = 1$. The proof is mere calculation. It's given that some integers a and b satisfy $ax + by = 1$. Thus

$$\begin{aligned} 1 &= (ax + by)^2 \\ &= a^2x^2 + b^2y^2 + 2abxy \\ &= a^2(x^2 + xy) + b^2(y^2 + xy) - (a^2 + b^2 - 2ab)xy \\ &= (a^2x + b^2y)(x + y) - (a - b)^2xy. \end{aligned}$$

A Boring Proof

Let's prove that if $(x, y) = 1$ then $(x + y, xy) = 1$. The proof is through the fundamental theorem of arithmetic. Suppose that the prime p divides xy. It must divide one of x and y, say x. As $(x, y) = 1$, it cannot divide y, hence cannot divide $x + y$.