Berge proved the following theorem: if \( \mathcal{B} \) is a field of sets then either \( \mathcal{B} \) or \( \mathcal{B} - \emptyset \) can be partitioned into pairs of disjoint sets. Since the paper is (apparently) not available online, we reproduce his proof here.

The proof is by induction on \(|\mathcal{B}|\). The base cases \( \mathcal{B} = \emptyset \) and \( \mathcal{B} = \{\emptyset\} \) are trivial, so suppose \(|\mathcal{B}| \geq 2\), and choose some \( x \) such that \( \{x\} \in \mathcal{B} \). We decompose \( \mathcal{B} \) into three sets: \( X = \{A \in \mathcal{B} : x \in A\} \), \( Y = \{A - x : A \in X\} \), \( Z = \mathcal{B} \setminus X \setminus Y \).

Note \( \emptyset \in Y \) and \(|Y \cup Z| < |\mathcal{B}|\). By the induction hypothesis, there is a matching \( M_1 \) on \( Y \) or \( Y - \emptyset \), and a matching \( M_2 \) on \( Y \cup Z \) or \( Y \cup Z - \emptyset \). We will construct a new matching \( M \) in which every element of \( \mathcal{B} \) is matched, except perhaps the empty set. The multigraph on \( Y \cup Z \) formed by taking the union of \( M_1 \) and \( M_2 \) consists of paths and cycles, which can be classified as follows:

- **Cycles** \( A_1, \ldots, A_\ell \) in which edges from \( M_1 \) and \( M_2 \) alternate. Note that \( A_i \in Y \) and \( \ell \) is even. The matching \( M \) includes \( \{A_1, A_2 + x\}, \{A_2, A_3 + x\}, \ldots, \{A_\ell, A_1 + x\} \).

- **Edges** \( A_1, A_2 \) taken from \( M_2 \). The matching \( M \) includes \( \{A_1, A_2\} \).

- **Paths** \( A_1, \ldots, A_\ell \) in which the edges alternate \( M_2 \) and \( M_1 \) and \( A_1 \in Z \). Note that \( A_2, \ldots, A_{\ell - 1} \in Y \) and either \( A_\ell \in Z \) or \( A_\ell = \emptyset \). In the former case, the matching \( M \) includes \( \{A_1, A_2 + x\}, \{A_2, A_3 + x\}, \ldots, \{A_{\ell - 2}, A_{\ell - 1} + x\}, \{A_{\ell - 1}, A_\ell\} \). In the latter case, the matching \( M \) includes \( \{A_1, A_2 + x\}, \{A_2, A_3 + x\}, \ldots, \{A_{\ell - 2}, A_{\ell - 1} + x\}, \{A_{\ell - 1}, A_\ell + x\} \), and \( A_\ell = \emptyset \) remains unmatched.