Berge's theorem

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Berge proved the following theorem: if $\mathcal{B}$ is a field of sets then either $\mathcal{B}$ or $\mathcal{B}-\emptyset$ can be partitioned into pairs of disjoint sets. Since the paper is (apparently) not available online, we reproduce his proof here.

The proof is by induction on $|\mathcal{B}|$. The base cases $\mathcal{B} = \emptyset$ and $\mathcal{B} = \{\emptyset\}$ are trivial, so suppose $|\mathcal{B}| \geq 2$, and choose some x such that $\{x\} \in \mathcal{B}$. We decompose $\mathcal{B}$ into three sets: $X = \{A \in \mathcal{B} : x \in A\}$, $Y = \{A - x : A \in X\}$, $Z = \mathcal{B} \setminus X \setminus Y$. Note $\emptyset \in Y$ and $|Y \cup Z| < |\mathcal{B}|$. By the induction hypothesis, there is a matching M_1 on Y or $Y - \emptyset$, and a matching M_2 on $Y \cup Z$ or $Y \cup Z - \emptyset$. We will construct a new matching M in which every element of $\mathcal{B}$ is matched, except perhaps the empty set. The multigraph on $Y \cup Z$ formed by taking the union of M_1 and M_2 consists of paths and cycles, which can be classified asfollows:

- Cycles $A_1, \ldots, A_\ell$ in which edges from M_1 and M_2 alternate. Note that $A_i \in Y$ and ℓ is even. The matching M includes $\{A_1, A_2 + x\}, \{A_2, A_3 + x\}, \ldots, \{A_\ell, A_1 + x\}$.
- Edges A_1, A_2 taken from M_2. The matching M includes $\{A_1, A_2\}$.
- Paths $A_1, \ldots, A_\ell$ in which the edges alternate M_2 and M_1 and $A_1 \in Z$. Note that $A_2, \ldots, A_{\ell-1} \in Y$ and either $A_\ell \in Z$ or $A_\ell = \emptyset$. In the former case, the matching M includes $\{A_1, A_2+x\}, \{A_2, A_3+x\}, \ldots, \{A_{\ell-2}, A_{\ell-1}+x\}, \{A_{\ell-1}, A_\ell\}$. In the latter case, the matching M includes $\{A_1, A_2 + x\}, \{A_2, A_3 + x\}, \ldots, \{A_{\ell-2}, A_{\ell-1} + x\}, \{A_{\ell-1}, A_\ell + x\}$, and $A_\ell = \emptyset$ remains unmatched.