Fourier analysis: KKL

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November 5, 2017

1. Another hypercontractive inequality:

   (a) Show that for all ρ we have $\langle T_\rho f, g\rangle = \langle f, T_\rho g\rangle$.

   (b) Deduce that $\|T_\rho f\|^2 = \langle f, T_{\rho^2} f\rangle$.

   (c) Hölder's inequality states that if $1/p + 1/q = 1$ for $p, q \geq 1$ (possibly $(p,q) = (1,\infty)$) then $\mathbb{E}[fg] \leq \|f\|_p \|g\|_q$. Use this to show that $\|T_{1/\sqrt{3}} f\|_2^2 \leq \|f\|_{4/3} \|T_{1/3} f\|_4$.

   (d) Last week we showed that $\|T_{1/\sqrt{3}} f\|_4 \leq \|f\|_2$ for all $f \colon \{\pm 1\}^n \to \mathbb{R}$. Deduce that

$$\|T_{1/\sqrt{3}} f\|_2 \leq \|f\|_{4/3}.$$

2. Spectral sample:
   Let $f \colon \{\pm 1\}^n \to \{\pm 1\}$.

   (a) Define a distribution $\mathcal{S}$ (the *spectral sample*) on subsets of $[n]$ by

$$\Pr[\mathcal{S} = S] = \hat{f}(S)^2.$$

   Why does this define a distribution?

   (b) Show that $\mathrm{Inf}[f] = \mathbb{E}_{S \sim \mathcal{S}}[|S|]$.

   (c) Show that $\langle f, T_\rho f\rangle = \mathbb{E}_{S \sim \mathcal{S}}[\rho^{|S|}]$.

   (d) Jensen's inequality states that if X is a random variable and φ is convex then $\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$. Use Jensen's inequality to show that for $\rho > 0$ we have

$$\langle f, T_\rho f\rangle \geq \rho^{\mathrm{Inf}[f]}.$$

3. Modified spectral sample:
   Let $f \colon \{\pm 1\}^n \to \{\pm 1\}$.

   (a) Define a distribution $\mathcal{S}'$ on *non-empty* subsets of $[n]$ by

$$\Pr[\mathcal{S}' = S] = \frac{\hat{f}(S)^2}{\mathbb{V}[f]}.$$

   Why does this define a distribution?

   (b) Show that for $\rho > 0$ we have

$$\sum_{S \neq \emptyset} \rho^{|S|} \hat{f}(S)^2 \geq \mathbb{V}[f] \rho^{\mathrm{Inf}[f]/\mathbb{V}[f]}.$$

4. Kahn–Kalai–Linial (KKL) theorem:
Let $f\colon \{\pm 1\}^n \to \{\pm 1\}$, and for $1 \le i \le n$ define
$$f_i(x) = \frac{f(x) - f(x^{\oplus i})}{2}.$$
Recall that $f_i = \sum_{i \in S} \hat{f}(S)\chi_S$ and $\operatorname{Inf}_i[f] = \|f_i\|_2^2$.
   (a) Show that $\|f_i\|_p^p$ is independent of p.
   (b) Use hypercontractivity to show that $\|T_{1/\sqrt{3}} f_i\|_2^2 \le \operatorname{Inf}_i[f]^{3/2}$.
   (c) Show that $\sum_{i=1}^n \|T_{1/\sqrt{3}} f_i\|_2^2 \ge \sum_{S \neq \emptyset} 3^{-|S|} \hat{f}(S)^2$.
   (d) Using the modified spectral sample, deduce that
$$\frac{1}{(\operatorname{Inf}[f]/\mathbb{V}[f]) 3^{\operatorname{Inf}[f]/\mathbb{V}[f]}} \le \sqrt{\max_i \operatorname{Inf}_i[f]}.$$
   (e) Show that if $\max_i \operatorname{Inf}_i[f] \le C \frac{\log_3 n}{n} \mathbb{V}[f]$ for some $C > 0$ then
$$\frac{1}{C \log_3 n \cdot n^C} \le \sqrt{C \frac{\log_3 n}{n} \mathbb{V}[f]}.$$
   (f) Deduce the KKL theorem:
$$\max_i \operatorname{Inf}_i[f] = \Omega\left(\frac{\log n}{n} \mathbb{V}[f]\right).$$

Exercises (optional):

1. Bribing:
Consider an election between two candidates, A and B, involving n voters.
The semantics of the election can be described by a function $\{A, B\}^n \to \{A, B\}$ called a *voting rule*.
We can assume that the voting rule is *monotone*: if a voter changes her opinion from B to A, then this can only be in A's favor.
We can also assume it is *fair*: if we switch A's and B's in the input, then this switches A's and B's in the output.
Show that for every $\epsilon > 0$ there exists a set of $O_\epsilon(n/\log n)$ voters such that if A bribes these voters to vote for her and the rest of the voters vote randomly (without any particular preference to A or B), then she wins with probability $1 - \epsilon$.

2. Tribes function:
   (a) A function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is called *transitive-symmetric* if for every i, j there exists a permutation π mapping i to j such that
$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n).$$
   Show that if f is transitive-symmetric then $\operatorname{Inf}[f] = \Omega(\mathbb{V}[f] \log n)$.
   (b) For a parameter m dividing n, the Tribes function is a function on n inputs given by
$$\mathrm{Tribes}_{n,m} = \min_{1 \le i \le n/m} \max_{1 \le j \le m} x_{i,j}.$$
   Show that $\mathrm{Tribes}_{n,m}$ is transitive-symmetric.
   (c) Find a value of m for which $\mathbb{E}[\mathrm{Tribes}_{n,m}] \approx 0$, and define $\mathrm{Tribes}_n = \mathrm{Tribes}_{n,m}$.
   (d) Estimate $\operatorname{Inf}[\mathrm{Tribes}_n]$, and deduce that $\operatorname{Inf}[\mathrm{Tribes}_n] = \Omega(\mathbb{V}[\mathrm{Tribes}_n] \log n)$.