Fourier analysis: Noise

Yuval Filmus

November 12, 2017

In the sequel, $x \sim \{-1,1\}^n$ will mean that x is chosen uniformly from $\{-1,1\}^n$.

1. Noise operator:

   Given $x \in \{-1,1\}^n$ and $\rho \in [-1,1]$, let $N_\rho(x)$ be the distribution given by

   $$y_i = \begin{cases} x_i & \text{with probability } \frac{1+\rho}{2}, \\ -x_i & \text{with probability } \frac{1-\rho}{2}. \end{cases}$$

   (a) Show that $\mathbb{E}[x_i y_i] = \rho$. (We say that x, y are ρ-*correlated*.)

   (b) Show that if $x \sim \{-1,1\}^n$ and $y \sim N_\rho(x)$ then $y \sim \{-1,1\}^n$.

   (c) Let x, y be chosen as in the previous item (we write $(x,y) \sim N_\rho$). Show that $x \sim N_\rho(y)$.

2. The *noise operator* is an operator on functions $\{-1,1\}^n \to \mathbb{R}$ given by

   $$(T_\rho f)(x) = \mathop{\mathbb{E}}_{y \sim N_\rho(x)}[f(y)].$$

   (a) Show that

   $$\langle f, T_\rho g \rangle = \mathop{\mathbb{E}}_{(x,y) \sim N_\rho}[f(x) g(y)].$$

   (b) Calculate that $T_\rho \chi_S = \rho^{|S|} \chi_S$.

   (c) Deduce that $T_\rho f = \sum_{d=0}^n \rho^d f^{=d}$.

3. The goal of the this exercise is to prove the following result, known as *hypercontractivity*:

   $$\|T_{1/\sqrt{3}} f\|_4 \leq \|f\|_2,$$

   where $\|f\|_p = \mathbb{E}[|f|^p]^{1/p}$ (so $\|f\| = \|f\|_2$). The proof is by induction. For brevity, we will use T for $T_{1/\sqrt{3}}$.

   (a) Prove the base case $n = 0$. From now on we assume that $n > 0$, and that the result is known for $n - 1$.

   (b) Write $f(x_1, \ldots, x_n) = x_n g(x_1, \ldots, x_{n-1}) + h(x_1, \ldots, x_{n-1})$. Show that

   $$Tf = \frac{x_n}{\sqrt{3}} Tg + Th.$$

(c) Show that

$$\mathbb{E}[(Tf)^4] = \frac{1}{9}\,\mathbb{E}[(Tg)^4] + 2\,\mathbb{E}[(Tg)^2(Th)^2] + \mathbb{E}[(Th)^4].$$

(d) Use Cauchy–Schwartz to show that

$$\mathbb{E}[(Tf)^4] \le \frac{1}{9}\,\mathbb{E}[(Tg)^4] + 2\sqrt{\mathbb{E}[(Tg)^4]\,\mathbb{E}[(Th)^4]} + \mathbb{E}[(Th)^4].$$

(e) Use the induction hypothesis to deduce

$$\mathbb{E}[(Tf)^4] \le \frac{1}{9}\,\mathbb{E}[g^2]^2 + 2\,\mathbb{E}[g^2]\,\mathbb{E}[h^2] + \mathbb{E}[h^2]^2 \le (\mathbb{E}[g^2] + \mathbb{E}[h^2])^2.$$

(f) Show that

$$\mathbb{E}[f^2] = \mathbb{E}[g^2] + \mathbb{E}[h^2]$$

and conclude the proof.

Homework: Application of hypercontractivity

1. Low-degree functions are reasonable:

   Suppose that $f\colon \{-1,1\}^n \to \mathbb{R}$ has degree d.

   (a) Show that $\|T_\rho f\|^2 = \sum_{e=0}^{d} \rho^{2e}\|f^{=e}\|^2$.

   (b) Deduce that $\|T_{\sqrt{3}} f\|^2 \le 3^d \|f\|^2$.

   (c) Show that $\|f\|_4 \le \sqrt{3}^d \|f\|$.

   (d) Concentration: Deduce that

   $$\Pr[|f - \mathbb{E}[f]| \ge s\sqrt{\mathbb{V}[f]}] \le \frac{9^d}{s^4}.$$

   How does this bound compare to Chebyshev's inequality? (Comment: more refined arguments give much better bounds on the tails of f.)

   (e) Prove (or look up) the *Paley–Zygmund inequality*: If $t \in [0,1]$ then

   $$\Pr[X > t\,\mathbb{E}[X]] \ge (1-t)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

   (f) Anticoncentration: Deduce that for $t \in [0,1]$,

   $$\Pr[|f - \mathbb{E}[f]| \ge t\sqrt{\mathbb{V}[f]}] \ge (1-t^2)^2 9^{-d}.$$

Bonus homework: Friedgut–Kalai–Naor (FKN) Our goal is to prove the following theorem (a version of the FKN theorem):

If $f\colon \{-1,1\}^n \to \{-1,1\}$ *satisfies* $\mathbb{E}[f] = 0$ *and* $\|f^{=1}\|^2 = 1-\epsilon$ *then* f *is* $O(\epsilon)$*-close to one of the functions* $\pm x_1, \ldots, \pm x_n$*, in the sense that* $\Pr[f \neq g] = O(\epsilon)$ *for some* $g = \pm x_i$*.*

(In general, we say that f, g are δ-close if $\mathbb{E}[(f-g)^2] \leq \delta$; this agrees with the preceding definition up to a multiplicative constant.)

Let f be as stated above.

1. Let $g = f^{=1}$. Show that $\mathbb{E}[g^2] = 1 - \epsilon$.
2. Show that there exists a constant $c > 0$ such that
$$\Pr[|g^2 - (1-\epsilon)| > \tfrac{1}{2}\sqrt{\mathbb{V}[g^2]}] \geq c.$$
3. Deduce that
$$\mathbb{E}[|g^2 - 1|] \geq c(\tfrac{1}{2}\sqrt{\mathbb{V}[g^2]} - \epsilon).$$
4. Let $h = f - g$. Show that $\mathbb{E}[h^2] = \epsilon$ and $g^2 - 1 = -2fh + h^2$.
5. Use the triangle inequality and Cauchy–Schwartz to show that
$$\mathbb{E}[|g^2 - 1|] \leq 2\sqrt{\epsilon} + \epsilon.$$
6. Combine both inequalities on $\mathbb{E}[|g^2 - 1|]$ to deduce that $\mathbb{V}[g^2] \leq A\epsilon$ for some $A > 0$.
7. Show that
$$\mathbb{V}[g^2] = 4 \sum_{1 \leq i < j \leq n} \hat{f}(\{i\})^2 \hat{f}(\{j\})^2.$$
8. Completing the square, deduce that
$$\mathbb{V}[g^2] = 2\big((1-\epsilon)^2 - \sum_{i=1}^{n} \hat{f}(\{i\})^4\big).$$
9. Deduce that $\max_i \hat{f}(\{i\})^2 \geq 1 - B\epsilon$ for some $B > 0$, and so $\max_i |\hat{f}(\{i\})| \geq 1 - B\epsilon$.
10. Deduce that for some $g \in \{\pm x_1, \ldots, \pm x_n\}$, $\Pr[f = g] \geq 1 - B\epsilon$.

Bonus question The Friedgut–Kalai–Naor is sometimes stated without the condition $\mathbb{E}[f] = 0$. What is the appropriate theorem in this case? State and prove!