Fourier analysis: Influence

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1. Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a Boolean function. The *ith influence* of f is
$$\mathrm{Inf}_i[f] = \Pr_{x \sim \{-1,1\}^n}[f(x) \neq f(x^{\oplus i})],$$
where $x^{\oplus i}$ is the point obtained from x by flipping the ith entry.
   (a) Show that $\mathrm{Inf}_i[f] = \frac{1}{4}\mathbb{E}[(f(x) - f(x^{\oplus i}))^2]$. This gives a definition of the influence for arbitrary (non-Boolean) functions.
   (b) Let
$$f_i(x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_{i-1},-1,x_{i+1},\ldots,x_n)}{2}.$$
Show that
$$\mathrm{Inf}_i[f] = \|f_i\|^2.$$
   (c) Define a function f by $f(x) = \chi_S(x^{\oplus i})$. What is f?
   (d) Show that
$$\mathrm{Inf}_i[f] = \sum_{S \ni i} \hat{f}(S)^2.$$
   (e) Suppose that f is a *monotone* Boolean function: if $x_i \leq y_i$ for $1 \leq i \leq n$ then $f(x_1,\ldots,x_n) \leq f(y_1,\ldots,y_n)$. Show that
$$\mathrm{Inf}_i[f] = \hat{f}(\{i\}) = \mathbb{E}[x_i f].$$

2. For a function $f\colon \{-1,1\}^n \to \mathbb{R}$, define the *total influence* of f by
$$\mathrm{Inf}[f] = \sum_{i=1}^n \mathrm{Inf}_i[f].$$
   (a) Give a formula for $\mathrm{Inf}[f]$ in terms of the Fourier expansion of f.
   (b) **Poincaré's inequality:** Deduce that $\mathbb{V}[f] \leq \mathrm{Inf}[f]$.
   (c) Suppose that f is Boolean (± 1-valued). The *sensitivity* of f at a point x is the number of neighbors y of x such that $f(y) \neq f(x)$ (x, y are neighbors if they differ by one coordinate). Show that the total influence of f is the same as the average sensitivity of f.

3. The *degree* of f is its degree as a multilinear polynomial. The *dth homogeneous part* of f is

$$f^{=d} = \sum_{|S|=d} \hat{f}(S)\chi_S.$$

We also define $f^{\leq d} = f^{=0} + \cdots + f^{=d}$ and $f^{>d} = f^{=d+1} + \cdots + f^{=n}$.

   (a) Show that $f = \sum_{d=0}^{n} f^{=d} = \sum_{d=0}^{\deg f} f^{=d}$.

   (b) Show that for all d, $f = f^{\leq d} + f^{>d}$.

   (c) Express $\|f^{=d}\|^2$ in terms of the Fourier coefficients of f, and use this to find a formula for $\mathrm{Inf}[f]$ in terms of $\|f^{=d}\|^2$.

   (d) Show that if $\deg f = d$ then $\mathrm{Inf}[f] \leq d\,\mathbb{V}[f]$.

4. A Boolean function $f \colon \{-1,1\}^n \to \{-1,1\}$ is *balanced* if $\mathbb{E}[f] = 0$.

   (a) Show that if f is a balanced Boolean function then its average sensitivity is at least 1.

   (b) Show that if f is a balanced Boolean function with average sensitivity 1 then $\deg f = 1$ and $f = f^{=1}$.

   (c) Deduce that if f is a balanced Boolean function with average sensitivity 1 then $f \in \{x_1, -x_1, \ldots, x_n, -x_n\}$.

Homework

1. Let $f \colon \{-1,1\}^n \to \{-1,1\}$ be a monotone function, where n is odd.

   (a) Show that $\mathrm{Inf}[f] = \mathbb{E}[(x_1 + \cdots + x_n)f]$.

   (b) Deduce that among all such functions f, the one maximizing the total influence is $f = \mathrm{sgn}(x_1 + \cdots + x_n)$ (usually known as *majority*).

   (c) Can you estimate the total influence of f?

Bonus homework: Boolean functions of degree d

1. Suppose that $g\colon \{0,1\}^n \to \mathbb{R}$.
   (a) Show that $g(y_1, \dots, y_n)$ can be represented uniquely as a multilinear polynomial in $y_1, \dots, y_n$.
   (b) Show that if g is integer-valued then the coefficients of its multilinear representation are integers.
2. Now suppose that $f\colon \{-1,1\}^n \to \{-1,1\}$ has degree $d \geq 1$.
   (a) Let $y_i = (1 + x_i)/2$, and define $g(y_1, \dots, y_n) = (1 + f(x_1, \dots, x_n))/2$. Relate the Fourier expansion of f to the multilinear expansion of g.
   (b) Deduce that $\hat{f}(S)$ is an integer multiple of 2^{1-d} for all S.
   (c) Using the triangle inequality, show that $|\hat{f}(\{i\})| \leq \mathbb{E}[|f_i|] = \mathrm{Inf}_i[f]$.
   (d) Deduce that for each i, either $\mathrm{Inf}_i[f] = 0$ or $\mathrm{Inf}_i[f] \geq 2^{1-d}$.
   (e) Show that $\mathrm{Inf}[f] \leq d$.
   (f) Deduce that f depends on at most $d2^{d-1}$ coordinates.
   (g) Construct a degree 2 Boolean function depending on 4 coordinates.
   (h) For each $d \geq 1$, construct a Boolean function depending on $2^d - 1$ coordinates (Hint: use a decision tree with d levels).
   (i) Combine the two constructions to obtain an improved lower bound on the number of coordinates that a degree d function can depend on.