Fourier analysis: Introduction

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1. Let \( f: \{-1,1\}^n \to \mathbb{R} \). (Note: sometimes \( \{-1,1\} \) is replaced by \( \{0,1\} \).)
   (a) Show that \( f(x_1, \ldots, x_n) \) can be written as a multilinear polynomial.
   (b) Show that the multilinear expansion is unique.

   The multilinear expansion of \( f \) is known as its Fourier expansion:
   \[
   f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S, \quad \text{where} \quad \chi_S = \prod_{i \in S} x_i.
   \]

   The coefficients \( \hat{f}(S) \) are known as the Fourier coefficients, and the functions \( \chi_S \) are known as the Fourier characters.

2. (a) Show that the Fourier characters constitute an orthonormal basis for \( \mathbb{R}[\{-1,1\}^n] \)
   with respect to the inner product
   \[
   \langle f, g \rangle = \mathbb{E}[fg] := 2^{-n} \sum_{x \in \{-1,1\}^n} f(x)g(x).
   \]
   (b) Show that \( \hat{f}(S) = \langle f, \chi_S \rangle \).
   (c) What is \( \hat{f}() \)?
   (d) Let \( \|f\|^2 := \langle f, f \rangle = \mathbb{E}[f^2] \). Show that \( \|f\|^2 = \sum_S \hat{f}(S)^2 \).
   (e) Parseval’s identity: Show that \( \sum_{S \neq \emptyset} \hat{f}(S)^2 = \mathbb{V}[f] \).

3. Suppose \( F \subseteq \{-1,1\}^n \). Let \( \mu(F) = |F|/2^n \). We can associate with \( F \) its indicator function \( f = 1_F \), given by \( f(x) = 1 \) if \( x \in F \) and \( f(x) = 0 \) if \( x \notin F \). A function whose range is \( \{0,1\} \) (or sometimes \( \{1,-1\} \)) is called Boolean.
   (a) Show that \( \hat{f}(\emptyset) = \mu(F) \).
   (b) Show that \( \sum_S \hat{f}(S)^2 = \mu(F) \).

4. Linearity testing: Let \( f: \{-1,1\}^n \to \{-1,1\} \).
   (a) Show that the Fourier characters satisfy \( \chi_S(xy) = \chi_S(x)\chi_S(y) \), where \( (xy)_i = x_i y_i \).

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\(^1\)All monomials are products of distinct variables, i.e., no monomial is divisible by \( x_i^2 \).
(b) Show that the Fourier characters satisfy $\chi_S(x)\chi_T(x) = \chi_{S \triangle T}(x)$, where $\triangle$ signifies symmetric difference.

c) Show that $E[\chi_S] = 1_{S = \emptyset}$.

d) Show that $\sum_S \hat{f}(S)^2 = 1$.

(e) Prove the following formula:
$$
\mathbb{E}_{x,y \sim \{-1,1\}^n} [f(x)f(y)f(xy)] = \sum_S \hat{f}(S)^3.
$$

(f) Prove that
$$
\mathbb{P}_{x,y \sim \{-1,1\}^n} [f(x)f(y) = f(xy)] = \frac{1}{2} + \frac{1}{2} \sum_S \hat{f}(S)^3.
$$

(g) Suppose that $\mathbb{P}[f(x)f(y) = f(xy)] \geq 1 - \epsilon$. Show that
$$
\max_S \hat{f}(S) \geq 1 - 2\epsilon.
$$

(h) Show that if $f, g$ are two $\{-1,1\}$-valued functions then
$$
\mathbb{P}[f = g] = \frac{1}{2} + \frac{1}{2} \mathbb{E}[fg].
$$

(i) Deduce that if $\mathbb{P}[f(x)f(y) = f(xy)] \geq 1 - \epsilon$ then there exists $S$ such that
$$
\mathbb{P}[f = \chi_S] \geq 1 - \epsilon.
$$

(j) Deduce that if $\mathbb{P}[f(x)f(y) = f(xy)] \geq \frac{1}{2} + \delta$ then there exists $S$ such that the correlation between $f$ and $\chi_S$ (i.e., $\mathbb{E}[f\chi_S]$) is at least $\delta$.

**Homework** Let $\Omega_d$ be the set of all $d$th roots of unity. Show that every function $f : \Omega_d^n \to \mathbb{R}$ has a unique expansion as a polynomial in which in every monomial, the degree of every variable is smaller than $d$.

**Challenge** Let $k \leq n/2$ and define
$$
\binom{[n]}{k} = \{ (x_1, \ldots, x_n) \in \{0,1\}^n : x_1 + \cdots + x_n = k \}.
$$

Show that every function $f : \binom{[n]}{k} \to \mathbb{R}$ has a unique expansion as a multilinear polynomial $P$ of degree at most $k$ satisfying
$$
\frac{\partial P}{\partial x_1} + \cdots + \frac{\partial P}{\partial x_n} = 0.
$$