Fourier analysis: Introduction

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1. Let $f\colon \{-1,1\}^n \to \mathbb{R}$. (Note: sometimes $\{-1,1\}$ is replaced by $\{0,1\}$.)

   (a) Show that $f(x_1,\ldots,x_n)$ can be written as a multilinear[1] polynomial.

   (b) Show that the multilinear expansion is *unique*.

   The multilinear expansion of f is known as its *Fourier expansion*:

   $$f(x_1,\ldots,x_n) = \sum_{S\subseteq[n]} \hat{f}(S)\chi_S, \text{ where } \chi_S = \prod_{i\in S} x_i.$$

   The coefficients $\hat{f}(S)$ are known as the *Fourier coefficients*, and the functions χ_S are known as the *Fourier characters*.

2. (a) Show that the Fourier characters constitute an orthonormal basis for $\mathbb{R}[\{-1,1\}^n]$ with respect to the inner product

   $$\langle f,g\rangle = \mathbb{E}[fg] := 2^{-n} \sum_{x\in\{-1,1\}^n} f(x)g(x).$$

   (b) Show that $\hat{f}(S) = \langle f,\chi_S\rangle$.

   (c) What is $\hat{f}(\emptyset)$?

   (d) Let $\|f\|^2 := \langle f,f\rangle = \mathbb{E}[f^2]$. Show that $\|f\|^2 = \sum_S \hat{f}(S)^2$.

   (e) **Parseval's identity:** Show that $\sum_{S\neq\emptyset} \hat{f}(S)^2 = \mathbb{V}[f]$.

3. Suppose $F \subseteq \{-1,1\}^n$. Let $\mu(F) = |F|/2^n$. We can associate with F its indicator function $f = 1_F$, given by $f(x) = 1$ if $x \in F$ and $f(x) = 0$ if $x \notin F$. A function whose range is $\{0,1\}$ (or sometimes $\{1,-1\}$) is called *Boolean*.

   (a) Show that $\hat{f}(\emptyset) = \mu(F)$.

   (b) Show that $\sum_S \hat{f}(S)^2 = \mu(F)$.

4. **Linearity testing:** Let $f\colon \{-1,1\}^n \to \{-1,1\}$.

   (a) Show that the Fourier characters satisfy $\chi_S(xy) = \chi_S(x)\chi_S(y)$, where $(xy)_i = x_i y_i$.

[1] All monomials are products of distinct variables, i.e., no monomial is divisible by x_i^2.

(b) Show that the Fourier characters satisfy $\chi_S(x)\chi_T(x) = \chi_{S\triangle T}(x)$, where $\triangle$ signifies symmetric difference.

(c) Show that $\mathbb{E}[\chi_S] = 1_{S=\emptyset}$.

(d) Show that $\sum_S \hat{f}(S)^2 = 1$.

(e) Prove the following formula:

$$\mathop{\mathbb{E}}_{x,y\sim\{-1,1\}^n}[f(x)f(y)f(xy)] = \sum_S \hat{f}(S)^3.$$

(f) Prove that

$$\Pr_{x,y\sim\{-1,1\}^n}[f(x)f(y) = f(xy)] = \frac{1}{2} + \frac{1}{2}\sum_S \hat{f}(S)^3.$$

(g) Suppose that $\Pr[f(x)f(y) = f(xy)] \geq 1-\epsilon$. Show that

$$\max_S \hat{f}(S) \geq 1-2\epsilon.$$

(h) Show that if f, g are two $\{-1,1\}$-valued functions then

$$\Pr[f = g] = \frac{1}{2} + \frac{1}{2}\mathbb{E}[fg].$$

(i) Deduce that if $\Pr[f(x)f(y) = f(xy)] \geq 1-\epsilon$ then there exists S such that

$$\Pr[f = \chi_S] \geq 1-\epsilon.$$

(j) Deduce that if $\Pr[f(x)f(y) = f(xy)] \geq \frac{1}{2}+\delta$ then there exists S such that the correlation between f and χ_S (i.e., $\mathbb{E}[f\chi_S]$) is at least δ.

Homework Let Ω_d be the set of all dth roots of unity. Show that every function $f\colon \Omega_d^n \to \mathbb{R}$ has a unique expansion as a polynomial in which in every monomial, the degree of every variable is smaller than d.

Challenge Let $k \leq n/2$ and define

$$\binom{[n]}{k} = \{(x_1,\ldots,x_n) \in \{0,1\}^n : x_1+\cdots+x_n = k\}.$$

Show that every function $f\colon \binom{[n]}{k} \to \mathbb{R}$ has a unique expansion as a multilinear polynomial P of degree at most k satisfying

$$\frac{\partial P}{\partial x_1} + \cdots + \frac{\partial P}{\partial x_n} = 0.$$