Complexity measures of functions

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January 7, 2018

Let $f\colon \{0,1\}^n \to \{0,1\}$. In Worksheet 2 and in Bonus 2 we defined the following two complexity measures:

- The degree of f, denoted $\deg(f)$, is the degree of the Fourier expansion of f.
- The approximate degree of f, denoted $\widetilde{\deg}(f)$, is the minimum degree of a polynomial g such that $|f(x) - g(x)| \leq 1/3$ for all $x \in \{0,1\}^n$ (changing the parameter $1/3$ to any other constant in the range $(0,1/2)$ affects $\widetilde{\deg}(f)$ by at most a constant factor).

Our goal in this bonus worksheet is to prove that $\deg(f) = O(\widetilde{\deg}(f)^6)$. On the way to this goal, we will need to discuss the following complexity measures:

- The *sensitivity* of f at a point x is the number of indices i such that $f(x) \neq f(x^{\oplus i})$. The sensitivity of f, denoted $s(f)$, is the maximum senitivity of f at any point (equivalently, it is the minimal s such that the sensitivity of f at all points is at most s).
- The *block sensitivity* of f at a point x is the maximum number of disjoint non-empty sets of indices S_i such that $f(x) \neq f(x^{\oplus S_i})$ for all i, where $x^{\oplus S_i}$ is obtained from x by flipping all coordinates in S_i. The block sensitivity of f, denoted $bs(f)$, is the maximum block sensitivity of f at any point.
- The *certificate complexity* of f at a point x is the minimum size of a set I of indices such that $f(x) = f(y)$ whenever $x_i = y_i$ for all $i \in I$ (the set I is known as a *certificate* or *witness*). The certificate complexity of f, denoted $C(f)$, is the maximum certificate complexity of f at any point.
- The (deterministic) *decision tree complexity* of f, denoted $D(f)$, is the minimum height (number of queries) of a decision tree computing f.

The rest of the worksheet is devoted to the proof of $\deg(f) = O(\widetilde{\deg}(f)^6)$, together with a non-matching lower bound.

1. Show that $s(f) \le bs(f) \le C(f)$.

2. **Upper bound on $C(f)$:**

   (a) Suppose that the block sensitivity of f at x is t. Show that there exist t disjoint non-empty sets $S_1, \dots, S_t$ such that for $i \in [t]$, we have (i) $f(x) \neq f(x^{\oplus S_i})$ and (ii) $f(x) = f(x^{\oplus T_i})$ for every proper subset T_i of S_i.

   (b) Show that the size of each of the sets S_i is at most $s(f)$.

   (c) Show that $S_1 \cup \dots \cup S_t$ is a certificate for f at the point x.

   (d) Conclude that $C(f) \le s(f)bs(f)$.

3. Show that $\deg(f) \le D(f)$.

4. **Upper bound on $D(f)$:** Consider the following algorithm, whose input is a point $x \in \{0,1\}^n$:

   - Initialize a vector $y \in \{0,1,\perp\}^n$ with $y_i := \perp$ for all i (here $\perp$ means "don't know").
   - Repeat $bs(f)$ times:
     - If $f(z) = 0$ for all points z consistent with y, then output 0.
     - Otherwise, find a point z consistent with y such that $f(z) = 1$.
     - Let I be a certificate for f at z.
     - Query x_i for $i \in I$, and update y accordingly: $y_i := x_i$ for all $i \in I$.
     - If $x_i = z_i$ for all $i \in I$, output 1.
   - Find a point z consistent with y, and output $f(z)$.

   This algorithm corresponds to a decision tree whose properties we now analyze.

   (a) Show that the algorithm can be implemented by a decision tree of height at most $C(f)bs(f)$.

   (b) Show that if the algorithm terminates inside the main loop, then it outputs $f(x)$.

   (c) Show that if the algorithm terminates at the final step, then it outputs $f(x)$. (Hint: suppose not, and consider two inputs z_0, z_1 consistent with y such that $f(z_0) = 0$ and $f(z_1) = 1$.)

   (d) Conclude that $\deg(f) \le D(f) \le C(f)bs(f) \le s(f)bs(f)^2 \le bs(f)^3$.

5. **Conclusion:**

   (a) Show that $bs(f) = O(\widetilde{\deg}(f)^2)$ by checking that Question 2(a)–(e) of Bonus 2 applies to this more general situation.

   (b) Conclude that $\deg(f) = O(\widetilde{\deg}(f)^6)$.

   (c) Show that some function f satisfies $\deg(f) = \Omega(\widetilde{\deg}(f)^2)$. This is the best known separation.

   (d) Show that the measures $\deg(f), \widetilde{\deg}(f), bs(f), C(f), D(f)$ are all polynomially related (any two measures $\alpha(f), \beta(f)$ satisfy $\alpha(f) = O(\beta(f)^{O(1)})$).

It is not known whether $s(f)$ is polynomially related to the other measures. The *sensitivity conjecture* states that it is. For more on this material, I suggest the paper *Separations in query complexity using cheat sheets* by Scott Aaronson, Shalev Ben-David, and Robin Kothari.