Chebyshev Polynomials

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Theory

1. Chebyshev polynomials.

(a) Show that for all $n \geq 0$ there exists a (unique) degree $n$ polynomial $T_n$ such that $T_n(\cos \theta) = \cos n\theta$ for all $\theta \in \mathbb{R}$.

(b) Show that $T_n$ is defined by the recurrence $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

(c) Show that $T_n(T_m(x)) = T_{nm}(x)$.

(d) Show that $|T_n(x)| \leq 1$ whenever $|x| \leq 1$.

(h) Show that the unique points in $[-1, 1]$ at which $|T_n(x)| = 1$ are $x_k = \cos(\pi k/n)$ for $0 \leq k \leq n$.

(i) Show that $|T_n'(1)| = n^2$. (More generally, $T_n^{(m)}(1) = n^m \frac{2^{m-1}}{1 \cdot 3 \cdot \cdots \cdot (2m-1)}$.)

(f) Show that $|T_n(x)| \leq 1$ whenever $|x| \leq 1$.

(g) Show that the unique points in $[-1, 1]$ at which $|T_n(x)| = 1$ are $x_k = \cos(\pi k/n)$ for $0 \leq k \leq n$.

(h) Denote by $c_n(P)$ the coefficient of $x^n$ in $P(x)$. Show that if $n \geq 1$ then $c_n(T_n) = 2^{n-1}$.

(e) Deduce that $c_n(S) = c_n(T_n)$.

2. Extremal property of top coefficient. Let $n \geq 1$.

Suppose that $S$ is a degree $n$ polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ and $c_n(S) > c_n(T_n)$.

(a) Let $R = \frac{c_n(S)}{c_n(T_n)} T_n - S$. Show that deg $R \leq n - 1$.

(b) Show that $\sgn R(x_k) = \sgn T_n(x_k)$ for $0 \leq k \leq n$.

(c) Deduce that $R$ has at least $n$ zeroes in $(-1, 1)$, and so $R = 0$.

(d) Derive a contradiction, and conclude that if $S$ is a degree $n$ polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ then $|c_n(S)| \leq |c_n(T_n)|$.

(e) Using similar arguments, show that if furthermore $|c_n(S)| = |c_n(T_n)|$ then $S(x) = \pm T_n(x)$.

3. Extremal property of $m$th derivatives outside $(-1, 1)$. Let $n \geq 1$, $0 \leq m \leq n$, and $y \geq 1$.

Suppose that $S$ is a degree $n$ polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ and $S^{(m)}(y) > T_n^{(m)}(y)$.

(a) Let $R = \frac{S^{(m)}(y)}{T_n^{(m)}(y)} T_n - S$. Show that deg $R \leq n$ and $\deg R^{(m)}(x) \leq n - m$.

(b) Show that $\sgn R(x_k) = \sgn T_n(x_k)$ for $0 \leq k \leq n$.

(c) Deduce that $R$ has at least $n$ zeroes in $(-1, 1)$, and so $R^{(m)}$ has at least $n - m$ zeroes in $(-1, 1)$.

(d) Show that $R^{(m)}(y) = 0$, and deduce that $R^{(m)} = 0$ and so $\deg R < m$.

(e) Deduce that $c_n(S) > c_n(T_n)$, and derive a contradiction.

Conclude that if $S$ is a degree $n$ polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ then $|S^{(m)}(y)| \leq |T_n^{(m)}(y)|$ for all $|y| \geq 1$ and $0 \leq m \leq n$. 


The Markov brothers’ inequality\(^1\) states that if \(S\) is a degree \(n\) polynomial such that \(|S(x)| \leq 1\) for all \(|x| \leq 1\) then \(|S^{(m)}(x)| \leq T_n^{(m)}(1)\) for all \(|x| \leq 1\).

**APPLICATIONS**

1. \(L_1\) influences.
   Let \(f: \{-1,1\}^n \rightarrow \{-1,1\}\), and identify \(f\) with the unique multilinear polynomial representing it. Define \(f_i(x_1, \ldots, x_n) = \left[ f(x_1, \ldots, x_n) - f(x_1, \ldots, -x_i, \ldots, x_n) \right] / (2x_i)\) (see also Worksheet 2).

   (a) Show that \(f_i = \partial f / \partial x_i\).

   (b) Show that if \(\deg f = d\) then \(\sum_{i=1}^n \|f_i\|^2 \leq d\). (Hint: use \(\inf_i |f| = \|f_i\|^\frac{2}{2}\))

   (c) Show that \(|f(x_1, \ldots, x_n)| \leq 1\) whenever \(|x_1|, \ldots, |x_n| \leq 1\).

   (d) Show that there exists a set \(S \subseteq [n]\) such that

   \[
   \sum_{i=1}^n |f_i(1, \ldots, 1)| = \sum_{i \in S} \frac{\partial f}{\partial x_i}(1, \ldots, 1) - \sum_{i \not\in S} \frac{\partial f}{\partial x_i}(1, \ldots, 1).
   \]

   (e) Show that each of the two summands is bounded in magnitude by \((\deg f)^2\).

   (f) Conclude that if \(\deg f = d\) then \(\sum_{i=1}^n |f_i| \leq 2d^2\), where \(\|h\|_1 = E[|h|]\).

2. Approximate degree of \(\text{OR}\).
   Let \(f: \{0,1\}^n \rightarrow \{0,1\}\). The approximate degree of \(f\), denoted \(\widetilde{\deg}(f)\), is the minimal degree of a polynomial \(p\) such that for all \(z \in \{0,1\}^n\) we have \(|f(z) - p(z)| \leq 1/3\).

   The \(n\)-variate \(\text{OR}\) function is given by \(\text{OR}_n(0, \ldots, 0) = 0\), and \(\text{OR}_n(z) = 1\) otherwise.

   (a) Suppose that \(|\text{OR}(z) - p(z)| \leq 1/3\) for all \(z \in \{0,1\}^n\). Define the symmetrization \(q(z)\) by

   \[
   q(z_1, \ldots, z_n) = \frac{1}{n!} \sum_{\pi \in S_n} p(z_{\pi(1)}, \ldots, z_{\pi(n)}).
   \]

   Show that there exists a univariate polynomial \(r\) of degree at most \(\deg p\) such that \(q(z_1, \ldots, z_n) = r(z_1 + \cdots + z_n)\) whenever \(z_1, \ldots, z_n \in \{0,1\}\).

   (b) Show that \(r(x) \geq 1/3\) for some \(0 \leq x \leq 1\).

   (c) Let \(c = \max_{0 \leq x \leq n} |r'(x)|\). Show that \(-1/3 - c/2 \leq r(x) \leq 4/3 + c/2\) for all \(0 \leq x \leq n\).

   (d) Massage \(r\) into a univariate polynomial \(s\) of degree \(\deg r\) such that \(|s(x)| \leq 1\) for \(|x| \leq 1\), and apply the Markov brothers’ inequality to deduce that \(|r'(x)| \leq (\deg r)^2\).

   (e) Use \(c \geq 1/3\) to deduce that \(\deg r \geq \sqrt{n/6}\), and so \(\widetilde{\deg}(\text{OR}_n) \geq \sqrt{n/6}\).

   (f) Show that \(T_k(y) \geq k^2\) for all \(y \geq 1\), and so \(T_k(1 + \delta) \geq 1 + k^2 \delta\) for \(\delta \geq 0\).

   (g) Massage \(T_k\) into a polynomial \(r\) of degree \(k\) such that \(|T_k(x)| \leq 1\) for \(|x| \leq 1\) translates to a bound on \(|r(x)|\) for \(1 \leq x \leq n\), and additionally \(r(0) = 2/3\).

   (h) Show that you can choose \(k = O(\sqrt{n})\) so that \(|r(x)| \leq 1/3\) for \(1 \leq x \leq n\).

   (i) Use \(p(x_1, \ldots, x_n) = r(x_1 + \cdots + x_n)\) to conclude that \(\widetilde{\deg}(\text{OR}_n) = \Theta(\sqrt{n})\).

   (j) Show that \(\widetilde{\deg}(\text{OR}_n) = \Theta(\sqrt{n})\) even if we replace \(1/3\) by any constant in \((0,1/2)\).

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\(^1\)The (much easier) case \(n = 1\) was proved by Andrey Markov, the namesake of Markov’s inequality, and the case \(n \geq 2\) was proved by his brother Vladimir, who tragically died at the age of 25 from tuberculosis.