Chebyshev Polynomials

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Theory

1. **Chebyshev polynomials.**
   (a) Show that for all $n \geq 0$ there exists a (unique) degree n polynomial T_n such that $T_n(\cos\theta) = \cos n\theta$ for all $\theta \in \mathbb{R}$.
   (b) Show that T_n is defined by the recurrence $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.
   (c) Show that $T_n(T_m(x)) = T_{nm}(x)$.
   (d) Show that $T_n(-x) = (-1)^n T_n(x)$.
   (e) Show that the polynomials $\{T_n : n \geq 0\}$ are orthogonal with respect to the inner product
   $$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)\,dx}{\sqrt{1-x^2}}.$$
   (f) Show that $|T_n(x)| \leq 1$ whenever $|x| \leq 1$.
   (g) Show that the unique points in $[-1, 1]$ at which $|T_n(x)| = 1$ are $x_k = \cos(\pi k/n)$ for $0 \leq k \leq n$, which satisfy $T_n(x_k) = (-1)^k$.
   (h) Denote by $c_n(P)$ the coefficient of x^n in $P(x)$. Show that if $n \geq 1$ then $c_n(T_n) = 2^{n-1}$.
   (i) Show that $T_n'(1) = n^2$. (More generally, $T_n^{(m)}(1) = \frac{n^2(n^2-1^2)\cdots(n^2-(m-1)^2)}{1\cdot 3\cdots\cdot(2m-1)}$.)

2. **Extremal property of top coefficient.** Let $n \geq 1$.
   Suppose that S is a degree n polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ and $c_n(S) > c_n(T_n)$.
   (a) Let $R = \frac{c_n(S)}{c_n(T_n)} T_n - S$. Show that $\deg R \leq n - 1$.
   (b) Show that $\operatorname{sgn} R(x_k) = \operatorname{sgn} T_n(x_k)$ for $0 \leq k \leq n$ (see 1g for the definition of x_k).
   (c) Deduce that R has at least n zeroes in $(-1, 1)$, and so $R = 0$.
   (d) Derive a contradiction, and conclude that if S is a degree n polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ then $|c_n(S)| \leq |c_n(T_n)|$.
   (e) Using similar arguments, show that if furthermore $|c_n(S)| = |c_n(T_n)|$ then $S(x) = \pm T_n(x)$.

3. **Extremal property of mth derivatives outside $(-1, 1)$.** Let $n \geq 1$, $0 \leq m \leq n$, and $y \geq 1$.
   Suppose that S is a degree n polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ and $S^{(m)}(y) > T_n^{(m)}(y)$.
   (a) Let $R = \frac{S^{(m)}(y)}{T_n^{(m)}(y)} T_n - S$. Show that $\deg R \leq n$ and $\deg R^{(m)}(x) \leq n - m$.
   (b) Show that $\operatorname{sgn} R(x_k) = \operatorname{sgn} T_n(x_k)$ for $0 \leq k \leq n$.
   (c) Deduce that R has at least n zeroes in $(-1, 1)$, and so $R^{(m)}$ has at least $n - m$ zeroes in $(-1, 1)$.
   (d) Show that $R^{(m)}(y) = 0$, and deduce that $R^{(m)} = 0$ and so $\deg R < m$.
   (e) Deduce that $c_n(S) > c_n(T_n)$, and derive a contradiction.
   Conclude that if S is a degree n polynomial such that $|S(x)| \leq 1$ for all $|x| \leq 1$ then $|S^{(m)}(y)| \leq |T_n^{(m)}(y)|$ for all $|y| \geq 1$ and $0 \leq m \leq n$.

The *Markov brothers' inequality*[1] states that if S is a degree n polynomial such that $|S(x)| \le 1$ for all $|x| \le 1$ then $|S^{(m)}(x)| \le T_n^{(m)}(1)$ for all $|x| \le 1$.

Applications

1. **L1 influences.**
Let $f\colon \{-1,1\}^n \to [-1,1]$, and identify f with the unique multilinear polynomial representing it. Define $f_i(x_1,\dots,x_n) = [f(x_1,\dots,x_n) - f(x_1,\dots,-x_i,\dots,x_n)]/(2x_i)$ (see also Worksheet 2).

   (a) Show that $f_i = \frac{\partial f}{\partial x_i}$.

   (b) Show that if $\deg f = d$ then $\sum_{i=1}^n \|f_i\|_2^2 \le d$. (Hint: use $\mathrm{Inf}_i[f] = \|f_i\|_2^2$.)

   (c) Show that $|f(x_1,\dots,x_n)| \le 1$ whenever $|x_1|,\dots,|x_n| \le 1$.

   (d) Show that there exists a set $S \subseteq [n]$ such that

$$\sum_{i=1}^n |f_i(1,\dots,1)| = \sum_{i\in S} \frac{\partial f}{\partial x_i}(1,\dots,1) - \sum_{i\in \overline{S}} \frac{\partial f}{\partial x_i}(1,\dots,1).$$

   (e) Show that each of the two summands is bounded in magnitude by $(\deg f)^2$.
(Hint: reduce to the case $m=1$, $y=1$ of the extremal property of mth derivatives.)

   (f) Conclude that if $\deg f = d$ then $\sum_{i=1}^n \|f_i\|_1 \le 2d^2$, where $\|h\|_1 = \mathbb{E}[|h|]$.

   (g) Show that there exists a degree d function f such that $\sum_{i=1}^n \|f_i\|_1 = d$.

The upper bound can be improved to d^2, but we believe that the true answer is d.

2. **Approximate degree of OR.**
Let $f\colon \{0,1\}^n \to \{0,1\}$. The approximate degree of f, denoted $\widetilde{\deg}(f)$, is the minimal degree of a polynomial p such that for all $z \in \{0,1\}^n$ we have $|f(z) - p(z)| \le 1/3$.
The n-variate OR function is given by $\mathrm{OR}_n(0,\dots,0) = 0$, and $\mathrm{OR}_n(z) = 1$ otherwise.

   (a) Suppose that $|\mathrm{OR}(z) - p(z)| \le 1/3$ for all $z \in \{0,1\}^n$. Define the symmetrization $q(z)$ by

$$q(z_1,\dots,z_n) = \frac{1}{n!}\sum_{\pi\in S_n} p(z_{\pi(1)},\dots,z_{\pi(n)}).$$

   Show that there exists a univariate polynomial r of degree at most $\deg p$ such that $q(z_1,\dots,z_n) = r(z_1+\cdots+z_n)$ whenever $z_1,\dots,z_n \in \{0,1\}$.

   (b) Show that $r'(x) \ge 1/3$ for some $0 \le x \le 1$.

   (c) Let $c = \max_{0\le x\le n} |r'(x)|$. Show that $-1/3 - c/2 \le r(x) \le 4/3 + c/2$ for all $0 \le x \le n$.

   (d) Massage r into a univariate polynomial s of degree $\deg r$ such that $|s(x)| \le 1$ for $|x| \le 1$, and apply the Markov brothers' inequality to deduce that $\frac{cn}{5/3+c} \le (\deg r)^2$.

   (e) Use $c \ge 1/3$ to deduce that $\deg r \ge \sqrt{n/6}$, and so $\widetilde{\deg}(\mathrm{OR}_n) \ge \sqrt{n/6}$.

   (f) Show that $T_k'(y) \ge k^2$ for all $y \ge 1$, and so $T_k(1+\delta) \ge 1 + k^2\delta$ for $\delta \ge 0$.

   (g) Massage T_k into a polynomial r of degree k such that $|T_k(x)| \le 1$ for $|x| \le 1$ translates to a bound on $|r(x)|$ for $1 \le x \le n$, and additionally $r(0) = 2/3$.

   (h) Show that you can choose $k = O(\sqrt{n})$ so that $|r(x)| \le 1/3$ for $1 \le x \le n$.

   (i) Use $p(x_1,\dots,x_n) = r(x_1+\cdots+x_n)$ to conclude that $\widetilde{\deg}(\mathrm{OR}_n) = \Theta(\sqrt{n})$.

   (j) Show that $\widetilde{\deg}(\mathrm{OR}_n) = \Theta(\sqrt{n})$ even if we replace $1/3$ by any constant in $(0,1/2)$.

[1] The (much easier) case $n=1$ was proved by Andrey Markov, the namesake of Markov's inequality, and the case $n \ge 2$ was proved by his brother Vladimir, who tragically died at the age of 25 from tuberculosis.