Random Graphs — Week 9

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1 Zero-one law

So far we have not encountered any property whose threshold is exactly $1/2$. That is, for the properties that we considered, the probability that $G(n, 1/2)$ satisfies the property tended to either 0 or 1. It is easy, however, to construct properties for which this is not the case. Here are some examples:

- The number of edges is between $n^2/4 - n$ and $n^2/4 + n$.
- The edge $(1, 2)$ exists.
- The number of edges is even.
- n is even.

We will describe a class of properties for which the limiting probability is necessarily either 0 or 1, namely the class of first-order properties. Let us start with some examples:

- There exists a triangle: $\exists x, y, z\ (x \sim y) \wedge (x \sim z) \wedge (y \sim z)$.
- There are no isolated vertices: $\forall x \exists y (x \sim y)$.
- Some vertex has degree at least 2: $\exists x, y, z\ (y \neq z) \wedge (x \sim y)(x \sim z)$.

All quantification is over vertices. We allow two relations: $x = y$ and $x \sim y$ (that is, x, y is an edge).

Our goal is to prove the following result:

Theorem 1. *Let φ be a first-order statement in the language of graphs. If φ holds for $G(\aleph_0, 1/2)$ then the probability that $G(n, 1/2)$ satisfies φ tends to 1, and if φ doesn't hold for $G(\aleph_0, 1/2)$ then the probability that $G(n, 1/2)$ satisfies φ tends to 0.*

Here $G(\aleph_0, 1/2)$ is the *countable random graph*, which we define below.

2 Extension axioms

The axiom $E_{k,\ell}$ states that if $x_1, \ldots, x_k, y_1, \ldots, y_\ell$ are $k + \ell$ different vertices then there exists a vertex $z \neq x_1, \ldots, x_k, y_1, \ldots, y_\ell$ connected to all x_i and to none of y_j. For each fixed k, ℓ, the probability that $G(n, 1/2)$ doesn't satisfy $E_{k,\ell}$ is at most

$$n^{k+\ell}(1 - 2^{-k-\ell})^{n-k-\ell} = O(n^{k+l}c_{k,\ell}^n) = o(1),$$

since $c_{k,\ell} = 1 - 2^{-k-\ell} < 1$. Hence $E_{k,\ell}$ holds with high probability.

We will show that the set of axioms $E_{k,\ell}$ is *complete*, in the sense that they determine the truth value of every first-order statement. This means that for each first-order statement, either φ or $\neg\varphi$ is provable from the axioms $E_{k,\ell}$. Since each proof mentions finitely many axioms, this implies that φ or $\neg\varphi$ (respectively) holds for $G(n, 1/2)$ with probability $1 - o(1)$, proving Theorem 1.

3 Countable random graph

In order to reason about the entire collection $E_{k,\ell}$, we need a model that satisfies all of them. Such a model has to be infinite. A simple example is the *countable random graph* $G(\aleph_0, 1/2)$, which is a random graph on the vertex set $\mathbb{N}$ in which each edge is present with probability 1/2. It is easy to check that each $E_{k,\ell}$ holds *with probability* 1 (taking the limit $n \to \infty$ in the formula above). Since there are only countably many statements $E_{k,\ell}$, with probability 1 *all* of them hold.

Let us say that a countable graph is *extendible* if it satisfies all axioms $E_{k,\ell}$. The argument above shows that extendible graphs exist. It is also not hard to construct one. Let $i|_j$ be the j'th bit of i. We construct a graph in which $i \sim j$ iff $i|_j = 1$ or $j|_i = 1$ (the condition has to be symmetric since the graph is undirected). Given $x_1, \ldots, x_k, y_1, \ldots, y_\ell$, let $z = 2^{x_1} + \cdots + 2^{x_k} + M$ for large M. By construction $z \sim x_i$ for all i. Also by construction, $z|_{y_j} = 0$, and for large enough M, $y_j|_z = 0$, and so $z \not\sim y_j$.

A crucial property of any two extendible graphs is that they are isomorphic. This is shown using the so-called *back-and-forth* argument. Enumerate the vertices of the first graph $a_1, a_2, \ldots$ and of the second graph $b_1, b_2, \ldots$. We construct an isomorphism π between the two graphs in infinitely many steps. At step $2i - 1$, we make sure that π mentions a_i, and at step $2i$, we make sure that π mentions b_i. More formally, at step t we have a partial isomorphism π_t, which mentions the first t vertices in the order $a_1, b_1, a_2, b_2, \ldots$. We ensure that $\pi_{t+1} \supseteq \pi_t$, and so $\bigcup_t \pi_t$ is an isomorphism between the two graphs.

Suppose that we are at step $2i - 1$, and have to handle a_i. If a_i is already handled by π_{2i-2}, then we let $\pi_{2i-1} = \pi_{2i-2}$. Otherwise, let X be the set of a-vertices mentioned by π_{2i-2} and connected to a_i, and let Y be the set of a-vertices mentioned by π_{2i-2} and not connected to a_i. Since the second graph is extendible, there exists a b-vertex connected to $\pi_{2i-2}(X)$, not connected to $\pi(Y)$, and different from all other b-vertices mentioned by π_{2i-2} (we can enforce this by adding them to $\pi_{2i-2}(X)$, say). We map this vertex to a_i to form π_{2i-1}.

4 Finishing the proof

To finish the proof, we need to show that the axioms $E_{k,\ell}$ determine the truth value of each statement φ. Suppose not. Then neither φ nor $\neg\varphi$ can be proved from the $E_{k,\ell}$. According to the completeness theorem, there are models of $E_{k,\ell} + \varphi$ and $E_{k,\ell} + \neg\varphi$. We can assume furthermore that these models are countable, by the downwards Löwnheim–Skolem theorem. Hence both models are isomorphic. But then it cannot be the case that φ holds in one but not the other! This contradiction shows that the truth value of φ must be determined by $E_{k,\ell}$.

A celebrated result of Shelah and Spencer extends the zero-one law to $G(n, 1/n^\alpha)$ for all irrational α, as well as to many other $G(n,p)$ models.