# Random Graphs — Week 8

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## 1 Semicircle law

The average deviation of the degree of a vertex from its mean $n/2$ is of order $\sqrt{n}$, but the worst deviation is of order $\sqrt{n \log n}$. We can improve on this using spectral methods.

Let $A$ be a matrix such that $A(i,i) = 1$, $A(i,j) = 1$ if there is an edge $(i,j)$, and $A(i,j) = -1$ if there is no edge $(i,j)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. Then

$$\sum_{i=1}^n \lambda_i^2 = \operatorname{Tr} A^2 = \sum_{i,j} A(i,j)^2 = n^2.$$

Therefore we can expect the magnitude of the eigenvalues to be roughly $\sqrt{n}$.

Wigner's celebrated *semicircle law* states that after normalizing by $\sqrt{n}$, the empirical distribution of eigenvalues tends to the distribution on $[-2,2]$ whose density is given by $\frac{\sqrt{4-t^2}}{2\pi}$ (this is just an upper semicircle of radius 2), in the sense that for any constant $-2 \leq \alpha \leq \beta \leq 2$, the number of eigenvalues in the range $[\alpha\sqrt{n}, \beta\sqrt{n}]$ is

$$n \int_\alpha^\beta \frac{\sqrt{4-t^2}}{2\pi}\, dt \pm o(n).$$

In particular, the number of eigenvalues whose magnitude is larger than, say, $3\sqrt{n}$ is $o(n)$. Compare this to the degree distribution, in which the standard deviation is roughly $\frac{1}{2}\sqrt{n}$, but the number of vertices whose degree is at least , say, $\frac{n}{2} + 100\sqrt{n}$ is $\Theta(n)$.

Füredi and Komlós showed[1] that with high probability, *all* eigenvalues of $A$ are at most $(2+\epsilon)\sqrt{n}$ in magnitude (for an arbitrary fixed $\epsilon > 0$). This suggests that planted cliques of size $\Theta(\sqrt{n})$ can be detected this way: if the graph contains a $k$-clique, then $\lambda_{\max}(A) \geq k$, since the indicator function $f$ of the clique satisfies

$$\frac{f' A f}{f' f} = \frac{k^2}{k} = k.$$

## 2 Lovász theta function

There are several ways of detecting cliques of size $\Theta(\sqrt{n})$. We will explain one which uses the Lovász theta function.

[1] Actually their proof has a mistake, but the mistake can be corrected

Let $G = (V, E)$ be an arbitrary graph, and let $A$ be a symmetric $V \times V$ matrix such that $A(i, j) = 1$ whenever $i = j$ or $(i, j) \in E$ (we call such matrices *legal* or *valid* for $G$). If $f$ is the indicator function of a clique of size $\ell$ then

$$\lambda_{\max}(A) \geq \frac{f' A f}{f' f} = \frac{\ell^2}{\ell} = \ell.$$

Therefore $\omega(A) \leq \lambda_{\max}(A)$. Furthermore, if equality holds then $f$ is an eigenvector of $A$, corresponding to an eigenvalue of $\ell$.

The Lovász theta function $\theta(G)$ is defined as the best upper bound obtainable this way. The theta function can be computed efficiently (up to an arbitrarily small error) using semidefinite programming.

## 2.1 Erdős–Ko–Rado theorem (bonus)

As an aside, we mention that many intersection theorems can be proved using the Lovász theta function. For example, let us sketch a proof of the Erdős–Ko–Rado theorem, which states that if $k \leq n/2$ then any subset of $\binom{[n]}{k}$ (all subsets of $\{1, \ldots, n\}$ of size $k$) in which any two sets intersect must have size at most $\binom{n-1}{k-1}$; this is achieved by the family of all sets containing some fixed element.

The *Kneser graph* is the graph on $\binom{[n]}{k}$ in which two sets are connected if they are disjoint. The degree of each vertex is $\binom{n-k}{k}$, and this is an eigenvalue of the graph (that is, of its adjacency matrix $A$), corresponding to the constant 1 vector. The other eigenvalues of the graph are $(-1)^d \binom{n-k-d}{k-d}$ for $1 \leq d \leq k$, with multiplicity $\binom{n}{d} - \binom{n}{d-1}$.

Now consider the matrix $J - cA$, where $J$ is the all 1s matrix. The constant 1 vector is an eigenvector of $J$ corresponding to the eigenvalue $\binom{n}{k}$, and all other eigenvalues are 0. Hence the eigenvalues of $J - cA$ are $\binom{n}{k} - c\binom{n-k}{k}$ and $(-1)^{d+1} c \binom{n-k-d}{k-d}$ for $1 \leq d \leq k$. The maximal eigenvalue is thus

$$\max\left(\binom{n}{k} - c\binom{n-k}{k}, c\binom{n-k-1}{k-1}\right).$$

The best choice of $c$ is the one that makes both of these equal, namely

$$c = \frac{\binom{n}{k}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} = \frac{\binom{n}{k}}{\left(1 + \frac{n-k}{k}\right)\binom{n-k-1}{k-1}} = \frac{\binom{n}{k}}{\frac{n}{k} \cdot \binom{n-k-1}{k-1}}.$$

The maximal eigenvalue is thus

$$\frac{\binom{n}{k}}{\frac{n}{k} \cdot \binom{n-k-1}{k-1}} \cdot \binom{n-k-1}{k-1} = \frac{k}{n} \cdot \binom{n}{k} = \binom{n-1}{k-1}.$$

Since $J - cA$ is a valid matrix for the *complement* of the Kneser graph, we deduce that any intersecting family (which is a clique in the complement of the Kneser graph) contains at most $\binom{n-1}{k-1}$ sets.

## 3 Feige–Krauthgamer algorithm

Clearly $\theta(G(n, 1/2, k)) \geq k$. It turns out that with high probability, $\theta(G(n, 1/2, k)) = k$. To show this, we exhibit a legal matrix $M$ satisfying $\lambda_{\max}(M) \leq k$.

Our starting point is the matrix $A$ in which $A(i, i) = 1$, $A(i, j) = 1$ if $(i, j)$ is an edge, and $A(i, j) = -1$ otherwise. If we arrange the vertices so that the clique vertices appear first, then the matrix looks as follows:

$$\begin{pmatrix} B & C' \\ C & D \end{pmatrix}$$

Here $B$ is a $k \times k$ matrix consisting only of 1s, $C$ is an $(n-k) \times k$ matrix in which each entry is $\pm 1$ with equal probability, and $D$ is an $(n-k) \times (n-k)$ symmetric square matrix with 1s on the diagonal, and all other entries (considered in pairs) are equally likely to be $\pm 1$.

Let $f$ be the characteristic vector of the clique. Then $Af \approx kf$. Indeed, if we write $f = (1\ 0)$ (the first part of length $k$, the second part of length $n-k$), then $Af = (k\ Cf)$. We would like to fix $A$ to a matrix $M$ such that $Mf = kf$.

Let $C_1, \ldots, C_{n-k}$ be the rows of $C$. Then $C_i f$ is simply the sum of the $i$th row of $f$. The simplest way to "kill" $Cf$ is to subtract from each of the $k$ entries in $C_i$ the value $(\sum_j C_{ij})/k$. However, we are not allowed to do this, since we cannot change the 1-entries of $C$. Instead, if there are $S_i$ entries equal to $-1$ in $C_i$ (so $\sum_j C_{ij} = -S_i + (k - S_i) = k - 2S_i$), then we subtract from each of them $(k - 2S_i)/S_i$.

To analyze the spectrum of $M$, let us generate $G(n, 1/2, k)$ in two steps: first generating a graph $G \sim G(n, 1/2)$, and then adding the clique. Let $U$ be the $A$-matrix which corresponds to $G$ (defined in the same way as above), let $V = A - U$, and let $W = M - A$. Let $\lambda_i(Q)$ be the $i$'th largest eigenvalue of the matrix $Q$. Then:

1. $U$ is a random symmetric sign matrix with 1s on the diagonal, so by Füredi–Komlós, with high probability $\lambda_1(U) \leq 3\sqrt{n}$.

2. $V$ is a random symmetric matrix with 0s on the diagonal, $0, 2$ entries chosen uniformly at random in the top left $k \times k$ corner, and zeroes elsewhere. To understand its spectrum, it suffices to consider only the top left $k \times k$ corner. Given a random sign matrix $V'$ with 1s on the diagonal, we can generate $V$ using the formula $V = J - V'$, where $J$ is the $k \times k$ all-1s matrix.

   According to Füredi–Komlós, with high probability $\lambda_1(-V') \leq 3\sqrt{k}$. Since $J$ has rank 1, linear algebra tells us that $\lambda_2(V) \leq 3\sqrt{k}$.

3. We bound $\lambda_1(W)$ using the identity $\lambda_1(W) \leq \sqrt{\mathrm{Tr}(W^2)}$, where $\mathrm{Tr}(W^2)$ is just the sum of entries. If $s_i = \sum_{j=1}^{k} C_{ij} \sim \mathrm{Bin}(k, 1/2)$, then $S_i = (k - s_i)/2$ and so

   $$\mathrm{Tr}(W^2) = 2 \sum_{i=1}^{n-k} S_i \cdot \left(\frac{s_i}{S_i}\right)^2 = 8 \sum_{i=1}^{n-k} \frac{s_i^2}{(k - s_i)^2}.$$

   The distribution of $s_i$ is concentrated around $k/2$, and in particular, a Chernoff bound shows that the probability that $s_i \notin [k/3, 2k/3]$ is at most $e^{-\Omega(k)}$. For $k = \omega(\log n)$, this shows that with high probability, each summand is $\Theta(k^2/k^2) = 1$, and so $\lambda_1(W) = O(\sqrt{n})$.

If $k = C\sqrt{n}$ for large enough constant $C$, then this shows that $\lambda_2(M) \leq \lambda_1(U) + \lambda_2(V) + \lambda_1(W) = O(\sqrt{n}) < k$. Since $Mf = kf$, it follows that $\lambda_1(M) = k$, and furthermore, the eigenspace of $k$ is spanned by $f$.

This shows that with high probability, $\theta(G(n, 1/2, k)) = k$. Furthermore, unless we are unlucky, we should be able to recover the clique from the first eigenvector. In general, we can recover the clique using the following observation: the probability that $\theta(G(n, 1/2, k)) = k$ is $1 - o(1/n)$. If we remove a vertex from the graph, we get a sample from either $G(n-1, 1/2, k)$ or $G(n-1, 1/2, k-1)$, depending on whether we removed a vertex of the planted clique or not. Using the theta function, we can distinguish between these two cases.