Random Graphs — Week 4

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1 Rest of the distribution

The Bonferroni inequalities can be extended to \( \Pr[X = t] \) for arbitrary \( t \). Here is an alternative route for analyzing this probability.

We know that with high probability, all triangles \( G(n, c/n) \) are vertex disjoint. This is because the density of the following two graphs is larger than 1: two triangles sharing a vertex, and two triangles sharing an edge. Hence roughly speaking, for the graph to contain \( t \) triangles, there need to be \( t \) disjoint triangles that it contains, the other \( n - 3t \) vertices supporting no triangles.

In order to formalize this idea, we will bound \( \Pr[X = t] \) from both directions, starting with the upper bound. If \( G \sim G(n, c/n) \) contains exactly \( t \) triangles, then either \( G \) contains two triangles which are non-vertex-disjoint (this happens with probability \( o(1) \)), or it contains \( t \) disjoint triangles, the other \( n - 3t \) vertices supporting no triangles. Therefore

\[
\Pr[X = t] \leq o(1) + \frac{1}{t!} \left( \binom{n}{3} \binom{n - 3}{3} \cdots \binom{n - 3(t - 1)}{3} \right) \left( \frac{c}{n} \right)^{3t} \Pr[G(n-3t, c/n) \text{ is triangle-free}].
\]

Carefully repeating our calculations in the preceding section, we see that the probability that \( G(n-3t, c/n) \) is triangle-free tends to \( e^{-c^3/6} \). Therefore the second summand is asymptotic to \( \frac{1}{t!} \left( \frac{c^3}{6} \right)^k e^{-c^3/6} \). In total,

\[
\Pr[X = t] \leq e^{-c^3/6} \frac{(c^3/6)^k}{k!} + o(1).
\]

For the lower bound, for every \( k \)-tuple \( \tau \) of vertex-disjoint triangles we construct an event \( E_\tau \) in which the only triangles are \( \tau \). Since the events are disjoint, it will follow that \( \Pr[X = t] \geq \sum_\tau \Pr[E_\tau] \). The event \( E_\tau \) is as follows:

1. The triangles in \( \tau \) belong to \( G \), and there are no other edges among these \( 3k \) vertices.
2. The graph on the remaining \( n - 3k \) vertices is triangle-free, and contains at most \( n \log n \) edges.
3. There are no other triangles in \( G \).

The probability that the first condition holds is

\[
(c/n)^{3k} (1 - c/n)^{\binom{k}{2} - 3k} = (1 - o(1))(c/n)^{3k}.
\]
The expected number of edges in $G(n - 3k, c/n)$ is at most $n^2 \cdot (c/n) = cn$, and so Markov’s inequality shows that it contains more than $n \log n$ edges with probability $o(1)$. Therefore the second condition holds with probability $e^{-c^3/6} - o(1)$.

Note that the first two conditions are independent, since the first one depends only on the edges among the $3k$ triangle vertices, and the second one depends only on the edges among the remaining $n - 3k$ vertices. The third condition will only depend on the remaining edges.

Now suppose that the first two conditions hold. There are two ways in which the graph can contain a triangle other than the ones in $\tau$. First, there could be a triangle sharing an edge with one of the triangles in $\tau$. There are $3k(n - 3k)$ such potential triangles, and each of them belongs to $G$ with probability $(c/n)^2$, and so this case happens with probability $O(1/n)$.

Second, there could be a triangle sharing a vertex with one of the triangles in $\tau$. Such a triangle should contain one of the $n \log n$ edges involving the remaining $n - 3k$ vertices. There are $3kn \log n$ such potential triangles, and each of them belongs to $G$ with probability $(c/n)^2$, so this case happens with probability $O(\log n/n)$. In total, assuming the first two conditions, the third one is met with probability $1 - o(1)$.

Summarizing,

$$\Pr[\mathcal{E}_\tau] = (1 - o(1))(c/n)^{3k} \cdot (e^{-c^3/6} - o(1)) \cdot (1 - o(1)) \sim e^{-c^3/6}(c/n)^{3k}.$$ 

There are $\frac{1}{k!}(\begin{pmatrix} n \\ 3 \end{pmatrix}) \cdots (\begin{pmatrix} n - 3(k-1) \\ 3 \end{pmatrix}) \sim \frac{1}{k!}(n^3/6)^k$ possible $\tau$, and so, since the events $\mathcal{E}_\tau$ are disjoint,

$$\Pr[X = k] \geq \sum_\tau \Pr[\mathcal{E}_\tau] \sim \frac{(n^3/6)^k}{k!} \cdot e^{-c^3/6}(c/n)^{3k} = e^{-c^3/6}(c^3/6)^k/k!.$$ 

Combining this with the upper bound, we conclude

$$\Pr[X = k] \to e^{-c^3/6}(c^3/6)^k/k!.$$ 

2 Where do we go from here?

One obvious question is generalizing the Poisson limit law to other graphs. Such a law doesn’t hold for all graphs, but it does hold for all strictly balanced graphs, which are graphs for which $m(H) = d(H)$ (balanced), and Moreover the maximum is achieved uniquely.

A different question is what happens when $p$ is above the threshold. When $np \to \infty$ and $n^2(1 - p) \to \infty$, the distribution of the number of triangles is roughly normal, in the sense that for every fixed $t$,

$$\Pr\left[\frac{X - \mathbb{E}[X]}{\sqrt{\mathbb{V}[X]}} < t\right] \to \Pr[N(0,1) < t].$$

One is also interested in large deviation properties. For each $\epsilon > 0$, the probability that $X < (1 - \epsilon)\mathbb{E}[X]$ or that $X > (1 + \epsilon)\mathbb{E}[X]$ behaves asymptotically as $e^{-Cn^2}$, where $C$ depends on $\epsilon$ and on the direction. The lower tail ($X < (1 - \epsilon)\mathbb{E}[X]$) is much easier to analyze than the upper tail.