Random Graphs — Week 4

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1 Rest of the distribution

The Bonferroni inequalities can be extended to $\Pr[X = t]$ for arbitrary t. Here is an alternative route for analyzing this probability.

We know that with high probability, all triangles $G(n, c/n)$ are vertex disjoint. This is because the density of the following two graphs is larger than 1: two triangles sharing a vertex, and two triangles sharing an edge. Hence roughly speaking, for the graph to contain t triangles, there need to be t disjoint triangles that it contains, the other $n - 3t$ vertices supporting no triangles.

In order to formalize this idea, we will bound $\Pr[X = t]$ from both directions, starting with the upper bound. If $G \sim G(n, c/n)$ contains exactly t triangles, then either G contains two triangles which are non-vertex-disjoint (this happens with probability $o(1)$), or it contains t disjoint triangles, the other $n-3t$ vertices supporting no triangles. Therefore

$$\Pr[X = t] \leq o(1) + \frac{1}{t!}\binom{n}{3}\binom{n-3}{3}\cdots\binom{n-3(t-1)}{3}\left(\frac{c}{n}\right)^{3t}\Pr[G(n-3t, c/n) \text{ is triangle-free}].$$

Carefully repeating our calculations in the preceding section, we see that the probability that $G(n - 3t, c/n)$ is triangle-free tends to $e^{-c^3/6}$. Therefore the second summand is asymptotic to $\frac{1}{k!}(c^3/6)^k e^{-c^3/6}$. In total,

$$\Pr[X = t] \leq e^{-c^3/6}\frac{(c^3/6)^k}{k!} + o(1).$$

For the lower bound, for every k-tuple τ of vertex-disjoint triangles we construct an event E_τ in which the only triangles are τ. Since the events are disjoint, it will follow that $\Pr[X = t] \geq \sum_\tau \Pr[E_\tau]$. The event E_τ is as follows:

1. The triangles in τ belong to G, and there are no other edges among these $3k$ vertices.
2. The graph on the remaining $n - 3k$ vertices is triangle-free, and contains at most $n \log n$ edges.
3. There are no other triangles in G.

The probability that the first condition holds is

$$(c/n)^{3k}(1 - c/n)^{\binom{k}{2}-3k} = (1 - o(1))(c/n)^{3k}.$$

The expected number of edges in $G(n-3k, c/n)$ is at most $n^2 \cdot (c/n) = cn$, and so Markov's inequality shows that it contains more than $n \log n$ edges with probability $o(1)$. Therefore the second condition holds with probability $e^{-c^3/6} - o(1)$.

Note that the first two conditions are independent, since the first one depends only on the edges among the $3k$ triangle vertices, and the second one depends only on the edges among the remaining $n-3k$ vertices. The third condition will only depend on the remaining edges.

Now suppose that the first two conditions hold. There are two ways in which the graph can contain a triangle other than the ones in τ. First, there could be a triangle sharing an edge with one of the triangles in τ. There are $3k(n-3k)$ such potential triangles, and each of them belongs to G with probability $(c/n)^2$, and so this case happens with probability $O(1/n)$.

Second, there could be a triangle sharing a vertex with one of the triangles in τ. Such a triangle should contain one of the $n \log n$ edges involving the remaining $n-3k$ vertices. There are $3kn \log n$ such potential triangles, and each of them belongs to G with probability $(c/n)^2$, so this case happens with probability $O(\log n/n)$. In total, assuming the first two conditions, the third one is met with probability $1-o(1)$.

Summarizing,

$$\Pr[E_\tau] = (1-o(1))(c/n)^{3k} \cdot (e^{-c^3/6} - o(1)) \cdot (1-o(1)) \sim e^{-c^3/6}(c/n)^{3k}.$$

There are $\frac{1}{k!}\binom{n}{3}\binom{n-3}{3}\cdots\binom{n-3(k-1)}{3} \sim \frac{1}{k!}(n^3/6)^k$ possible τ, and so, since the events E_τ are disjoint,

$$\Pr[X=k] \geq \sum_\tau \Pr[E_\tau] \sim \frac{(n^3/6)^k}{k!} \cdot e^{-c^3/6}(c/n)^{3k} = e^{-c^3/6}\frac{(c^3/6)^k}{k!}.$$

Combining this with the upper bound, we conclude

$$\Pr[X=k] \longrightarrow e^{-c^3/6}\frac{(c^3/6)^k}{k!}.$$

2 Where do we go from here?

One obvious question is generalizing the Poisson limit law to other graphs. Such a law doesn't hold for all graphs, but it does hold for all *strictly balanced* graphs, which are graphs for which $m(H) = d(H)$ (*balanced*), and moreover the maximum is achieved uniquely.

A different question is what happens when p is above the threshold. When $np \to \infty$ and $n^2(1-p) \to \infty$, the distribution of the number of triangles is roughly normal, in the sense that for every fixed t,

$$\Pr\left[\frac{X - \mathbb{E}[X]}{\sqrt{\mathbb{V}[X]}} < t\right] \to \Pr[N(0,1) < t].$$

One is also interested in large deviation properties. For each $\epsilon > 0$, the probability that $X < (1-\epsilon)\,\mathbb{E}[X]$ or that $X > (1+\epsilon)\,\mathbb{E}[X]$ behaves asymptotically as e^{-Cn^2}, where C depends on ϵ and on the direction. The lower tail ($X < (1-\epsilon)\,\mathbb{E}[X]$) is much easier to analyze than the upper tail.