Random Graphs — Assignment 1

Yuval Filmus

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Question 1. Fix a graph H, and suppose that $p = \omega(n^{-1/m(H)})$. In class, we showed that with high probability, $G(n,p)$ contains a copy of H. Show that in fact, with high probability $G(n,p)$ contains $\omega(1)$ copies of H.

Answer. Let X be the number of copies of H in $G(n,p)$. In class we showed that $\mathbb{E}[X] = \omega(1)$ and $\mathbb{V}[X] = o(\mathbb{E}[X]^2)$. Hence Chebyshev's inequality shows that

$$\Pr[X \le \mathbb{E}[X]/2] \le \Pr[|X - \mathbb{E}[X]| \ge \mathbb{E}[X]/2] \le \frac{\mathbb{V}[X]}{\mathbb{E}[X]^2/4} = o(1).$$

It follows that with high probability, $X \ge \mathbb{E}[X]/2 = \omega(1)$. □

Question 2. Suppose that $p = o(1/n)$. Estimate the probability that $G(n,p)$ contains a triangle by finding a function $q(n,p)$ such that

$$\Pr[G(n,p) \text{ contains a triangle}] \sim q(n,p).$$

(Note that as shown in class, $q(n,p) = o(1)$.)

Answer. Let X be the number of triangles in $G(n,p)$. The Bonferroni inequalities give the bounds

$$\mathbb{E}[X] - \mathbb{E}\left[\binom{X}{2}\right] \le \Pr[X > 0] \le \mathbb{E}[X].$$

In class we have shown that

$$\mathbb{E}[X] \sim (np)^3/6,$$

$$\mathbb{E}\left[\binom{X}{2}\right] = O(n^6p^6 + n^5p^6 + n^4p^5) = \mathbb{E}[X] \cdot O(n^3p^3 + n^2p^3 + np^2) = o(\mathbb{E}[X]).$$

It follows that

$$\Pr[X > 0] \sim \mathbb{E}[X] \sim (np)^3/6.$$

□

Question 3. In this question we will analyze the probability that $G(n, c/n)$ contains the graph $H := \rhd\!\!-$ (a triangle with an attached edge).

(a) Let E_k be the event that $G(n, c/n)$ contains exactly k vertex-disjoint triangles, each of them isolated (not connected by an edge to the outside world), and no other triangles. Calculate $\lim_{n\to\infty} \Pr[E_k]$, and deduce a lower bound on the probability that $G(n, c/n)$ is H-free (contains no copy of H).

Proof. For every specific choice of k vertex-disjoint triangles, the probability of the event is

$$(c/n)^{3k}(1-c/n)^{3k(n-3k)+\binom{k}{2}-3k}(e^{-c^3/6} \pm o(1)) \sim e^{-c^3/6}(c/n)^{3k}e^{-3ck}.$$

The number of possible choices is $\frac{1}{k!}\binom{n}{3}\cdots\binom{n-3(k-1)}{3} \sim \frac{1}{k!}(n^3/6)^k$, and so

$$\Pr[E_k] \sim e^{-c^3/6} \cdot e^{-3ck}\frac{(c^3/6)^k}{k!}.$$

Let us denote the right-hand side by q_k. We can calculate

$$q := \sum_{k=0}^{\infty} q_k = e^{-c^3/6}\sum_{k=0}^{\infty}\frac{((ce^{-c})^3/6)^k}{k!} = e^{-c^3/6} \cdot e^{(ce^{-c})^3/6} = e^{-(1-e^{-3c})(c^3/6)}.$$

For every $\epsilon > 0$, we can find ℓ such that $\sum_{k=0}^{\ell} q_\ell \geq q - \epsilon$. Hence for every $\epsilon > 0$, the probability that $G(n, c/p)$ is H-free is at least $q - \epsilon - o(1)$. It follows that the probability that $G(n, c/p)$ is H-free is at least $q - o(1)$. □

(b) Show that for every $\epsilon > 0$ there is k such that for large enough n, the probability that $G(n, c/n)$ contains more than k triangles is less than ϵ.

Answer. We know that for every k, the probability that $G(n, c/n)$ contains more than k triangles tends to

$$1 - e^{-c^3/6}\sum_{\ell=0}^{k}\frac{(c^3/6)^\ell}{\ell!}.$$

We can find k so that this is at most $\epsilon/2$. Hence for large enough n, the probability that $G(n, c/n)$ contains more than k triangles is less than ϵ. □

(c) Prove a matching upper bound on the probability that $G(n, c/n)$ is H-free.

Answer. If $G \sim G(n, c/n)$ is H-free then either one of the events $E_0, \ldots, E_k$ happens, or G contains more than k triangles. Note that G cannot contain two overlapping triangles, since any such configuration contains a copy of H. The calculation above shows that

$$\lim_{n\to\infty}\sum_{\ell=0}^{k}\Pr[E_\ell] < q,$$

and in particular, for large enough n the sum is at most q. The previous item shows that for every $\epsilon > 0$, we can find k such that the probability that G contains more than k triangles is at most ϵ. In total, we get that for every $\epsilon > 0$, for large enough n the probability that $G(n, c/n)$ is H-free is at most $q + \epsilon$. Since this holds for every $\epsilon > 0$, it follows that the probability that $G(n, c/n)$ is H-free is at most $q + o(1)$. □