Assignment 1: Solution Sketch

Itay Hazan
Random Graphs, Technioin, 2017

December 2, 2017

1 Emptyness Threshold

Exercise 1.a Let $X \sim \text{Bin}\left(\binom{n}{2}, p\right)$ denote the number of edges in the graph. Using Markov's inequality,

$$\Pr[G \text{ is not empty}] = \Pr[X \geq 1] \leq \mathbb{E}[X] = \binom{n}{2} p = o(1),$$

where the last equality holds since $\binom{n}{2} = \Theta(n^2)$ and $p = o\left(1/n^2\right)$.

Exercise 1.b Let X be defined as in 1.a. Using Chebyshev's inequality,

$$\Pr[G \text{ is empty}] = \Pr[X = 0] = \Pr[X - \mathbb{E}[X] = -\mathbb{E}[X]] \leq$$
$$\Pr[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]] \leq \frac{\mathbb{V}[X]}{\mathbb{E}[X]^2} = \frac{\binom{n}{2} p(1-p)}{\left(\binom{n}{2} p\right)^2} \leq \frac{1}{\binom{n}{2} p} = o(1),$$

and hence the graph is not empty whp.

Exercise 1.c In this case, the probability that the graph is empty is

$$\Pr[X = 0] = \left(1 - \frac{c}{n^2}\right)^{\binom{n}{2}} = \left(1 - \frac{c}{n^2}\right)^{\frac{n(n-1)}{2}} \longrightarrow e^{-c/2},$$

and therefore the graph is neither empty whp nor is it not empty whp.

2 Matching Threshold

Exercise 2.a Observe that a graph G is not a matching iff there exists a path length two in G. Let N be the number of such paths. In this exrecise (2.a) we shall prove that $N = 0$ whp, and hence the graph is a matching whp. To do so, given any three vertices u, v, w, let $\mathbf{1}_{u,v,w}$

be an indicator to the event "The path $u - v - w$ exists in G", and let $X = \sum_{u \neq v \neq w} \mathbf{1}_{u,v,w}$. Note that $X = 2N$, since X counts each 2-edge path twice (counting both $\mathbf{1}_{u,v,w}$ and $\mathbf{1}_{w,v,u}$ as equal to 1 if the undirected path $u - v - w$ exists in G). Nevertheless, we are dealing with asymptotics.

$$\mathbb{E}[X] = \sum_{u \neq v \neq w} \mathbb{E}\left[\mathbf{1}_{u,v,w}\right] = \binom{n}{3} p^2 = o(1),$$

and hence, using Markov's inequality, $\Pr[X > 0] \longrightarrow 0$, i.e. the graph is a matching whp.

Exercise 2.b Partition V into three disjoing equal-sized sets, $V = U \cup W \cup Q$, and let

$$Y = \sum_{u \in U} \sum_{w \in W} \sum_{q \in Q} \mathbf{1}_{u,w,q}.$$

Clearly, if $Y \geq 1$ then there exists some path of length two in the graph,and hence the graph is not a matching. It therefore suffices to claim that $Y > 0$ whp, which will be done using Chebyshev's inequality.

By linearity of expectation, $\mathbb{E}[Y] = \left(\frac{n}{3}\right)^3 p^2$. We now need to evaluate $\mathbb{V}[Y]$, which requires evaluating $\mathbb{E}[Y^2] = \sum_{u,w,q} \sum_{u',w',q'} \mathbb{E}[\mathbf{1}_{u,w,q} \mathbf{1}_{u',w',q'}]$. To do so,we separate into cases[1]:

- If $(u, w, q) = (u', w', q')$ then $\mathbb{E}[\mathbf{1}_{u,w,q} \mathbf{1}_{u',w',q'}] = \mathbb{E}[\mathbf{1}_{u,w,q}] = p^2$, and there are $\left(\frac{n}{3}\right)^3$ such summands in $\mathbb{E}[Y^2]$.
- If $(u, w) \neq (u', w')$ and $(w, q) \neq (w', q')$ then the paths are disjoint, and therefore $\mathbb{E}[\mathbf{1}_{u,w,q} \mathbf{1}_{u',w',q'}] = p^4$, and there are $\left(\frac{n}{3}\right)^3 \cdot \left(\left(\frac{n}{3}\right)^3 - 2 \cdot \frac{n}{3} + 1\right)$ such summands in $\mathbb{E}[Y^2]$ (there are $\left(\frac{n}{3}\right)^3$ choices for the first path,and once it was chosen, the second path can be any path, but: a path that has the same first edge and a different second edge (there are $\frac{n}{3} - 1$ such paths), a path that has the same second edge and a different first edge (there are $\frac{n}{3} - 1$ such paths), and the first path itself).
- If $(u, w) = (u', w')$ and $(w, q) \neq (w', q')$, then the paths share the first edge, and therefore $\mathbb{E}[\mathbf{1}_{u,w,q} \mathbf{1}_{u',w',q'}] = p^3$, and there are $\left(\frac{n}{3}\right)^4$ such summands in $\mathbb{E}[Y^2]$. The case where $(u, w) \neq (u', w')$ and $(w, q) = (w', q')$ (that is, the paths share the second edge) is similar.

Therefore,

$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \left(\frac{n}{3}\right)^3 p^2 + \left(\frac{n}{3}\right)^3 \cdot \left(\left(\frac{n}{3}\right)^3 - 2 \cdot \frac{n}{3} + 1\right) p^4 + 2\left(\frac{n}{3}\right)^4 p^3 - \left(\frac{n}{3}\right)^6 p^4,$$

[1]The reason we turn to Y, instead of working with X, is that $\mathbb{E}[X^2]$ requires considering more cases than $\mathbb{E}[Y^2]$ (e.g. triangles), which makes the argument a bit messier.

and, using Chebyshev's inequality, we thus conclude that

$$\Pr[Y = 0] \le \Pr[|Y - \mathbb{E}[Y]| \ge \mathbb{E}[Y]] \le \frac{\mathbb{V}[Y]}{\mathbb{E}[Y]^2} =$$
$$\frac{\left(\frac{n}{3}\right)^3 p^2 + \left(\frac{n}{3}\right)^3 \left(1 - 2 \cdot \frac{n}{3}\right) p^4 + 2\left(\frac{n}{3}\right)^4 p^3}{\left(\frac{n}{3}\right)^6 p^4} = \frac{1}{\left(\frac{n}{3}\right)^3 p^2} + \frac{1 - 2 \cdot \frac{n}{3}}{\left(\frac{n}{3}\right)^3} + \frac{2}{\left(\frac{n}{3}\right)^2 p} \longrightarrow 0.$$

3 The Countable Random Graph

Exercise 3.a Let $A, B \subseteq U$ be disjoint finite sets. We want to find an element s connected to all elements of A and not connected to any element of B. According to the first axiom above, there exists $b \in U$ such that $x \,\varepsilon\, b$ iff $x \in B$. Applying the first axiom to $A \cup \{b\}$, there exists $s \in U$ such that $x \,\varepsilon\, s$ iff $x \in A$ or $x = b$. Clearly s is connected to all elements of A.

Suppose, for the sake of contradiction, that s is connected to some element $\beta \in B$. Then either $\beta \,\varepsilon\, s$ or $s \,\varepsilon\, \beta$. If $\beta \,\varepsilon\, s$ then since A, B are disjoint, necessarily $\beta = b$. However, this implies that $b \,\varepsilon\, b$, contradicting the second axiom. If $s \,\varepsilon\, \beta$ then $b \,\varepsilon\, s \,\varepsilon\, \beta \,\varepsilon\, b$, again contradicting the second axiom.

Exercise 3.b For the first axiom, let $S \subset U$ be a finite set of natural numbers, and define $s = \sum_{s \in S} 2^s$. It is easy to see that the ith bit of s is 1 iff $i \in S$, and hence $i \in S$ iff $i \,\varepsilon\, s$.

The second axiom can be proved by induction. For the base case, let $x \,\varepsilon\, y$. This means that the xth bit of y is 1, and hence $y \ge 2^x$. Since $2^x > x$ for any $x \in \mathbb{N}$, then $y > x$, and hence $y \ne x$, as desired. The inductive step follows in a similar manner.