Assignment 0: Problem Set on Probability Theory

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No due date

1. **Penney's Game.**

   (a) Penney flips a fair coin twice. What is the probability that he gets two Heads (HH)? What is the probability that he get Heads followed by Tails (HT)? Are these probabilities the same?

   (b) Now, Penney flips a fair coin repeatedly until he gets two Heads in a row (HH). What is the expected number of coin flips it should take? What if Penney flips the coin until he gets Heads followed by Tails (HT)? Are these expectations the same?

   (c) Penney and his friend, Victor, now play the following game: they flip a coin repeatedly until either HH emerges (in which case, Penney wins) or HT emerges (Victor wins). Is the game fair?

   (d) Penney and Victor alter their game as follows: each now picks a three-outcome-pattern (i.e. $HHH, HHT, HTH, \dots$), and they flip a coin until either pattern emerges (in which case, the player who chose that pattern wins). Penney picks HHT as his pattern, and Victor picks THH. Is the game fair?

2. **The Two Children Problem.**

   (a) Alice has two children, one of whom is a girl. What is the probability that Alice has two girls?

   (b) Bob has two children, one of whom is a girl *born on a Tuesday*. What is the probability that Bob has two girls? (*Hint:* The answer is not the same)

3. **Elchanan Mossel's Dice Paradox.** You throw a die until you get 6. What is the expected number of throws (including the throw giving 6) **conditioned** on the event that all throws gave even numbers? (*Hint:* The answer is not 3)

4. **The Monty Hall Problem.** You have been selected to participate in the game show "It's All Monty Hall", where you are presented 3 doors. Monty Hall (†2017), the game show host, tells you that behind one of the doors stands a car, and behind each of the other two stands a goat. You pick one of the doors (at random, of course). Monty Hall now opens one of the two remaining doors, behind which stands a goat. You are then given the choice of either sticking with your door or switching to the second unopened door. What option should you choose?

5. **The Coupon Collector's Problem.** There are n different types of coupons. Each day, the mail carrier puts one (uniformly chosen) random coupon in your mailbox. Since there are many houses in the neighborhood, she cannot remember which coupon she already gave to whom, and thus the mail carrier may give the same coupon more than once. What is the expected number of days before you collect at least one copy of each of the n coupons? Give a tight asymptotic bound as a function of n.

6. **The Birthday Paradox.**

   (a) What is the minimal number of people n such that with probability at least $1/2$ there exists a pair that shares the same birthday?

   (b) Suppose now there are D days in a year. Give a tight asymptotic bound for n as a function of D.

7. **Fixed Points in Random Permutations.** A *permutation* on n elements is a bijection $\pi\colon [n] \to [n]$. A *fixed point* in a permutation π is an index $i \in [n]$ for which $\pi(i) = i$.

   (a) A *rearrangement* is a permutation that has no fixed points. Show that the number of rearrangements on n elements is approximately $n!/e$.

   (b) Prove that the expected number of fixed points in a (uniformly chosen) random permutation on n elements is 1.

   (c) Prove that as $n \to \infty$, the distribution of the number of fixed points in a random permutation on n elements approaches a Poisson distribution with rate $\lambda = 1$.

8. **Biased Random Walk on $\mathbb{Z}$.** Given $p \in [0,1]$, let $\left\{X_t^{(p)}\right\}_{t\geq 0}$ be a random walk on $\mathbb{Z}$ starting at 0, where the probability of jumping to the right is p and the probability of jumping to the left is $1-p$ (i.e., $X_0^{(p)} = 0$, $X_{t+1}^{(p)} = X_t^{(p)} + 1$ with probability p, and $X_{t+1}^{(p)} = X_t^{(p)} - 1$ with probability $1-p$).

   Assume $0 \leq q < r \leq 1$. Prove that for all $t \in \mathbb{N}$ and for all $z \in \mathbb{Z}$

   $$\Pr\left[X_t^{(q)} \leq z\right] \geq \Pr\left[X_t^{(r)} \leq z\right].$$

   (*Hint*: Use **coupling** of Bernoulli variables to define another pair of random walks on $\mathbb{Z}$, $\tilde{X}_t^{(q)}$ and $\tilde{X}_t^{(r)}$, and prove the inequality on the coupled pair)