Random Graphs: Assignment 3

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I have covered many different topics in this course, but there are many other possible topics. Here are some examples:

- Random CSPs and their solution spaces.
- Zero-one laws.
- Hamiltonicity (Chapter 6 of the textbook).
- Random graphs with weights on the edges (Chapter 18 of the textbook).
- Preferential attachment graphs (Chapter 17 of the textbook).
- Random lifts (especially recent work of Marcus–Spielman–Srivastava).

Please choose one topic and mark it in this Google document. Don't choose a topic which is already taken. You can discuss possible topics with me (by appointment).

Your task is to plan a 2-hour lecture on the topic, which I might use in a future instance of class, and submit it in PDF format. The plan should include:

- A bibliography of primary sources.
- A list of definitions and results that would be covered in class. Try to choose a selection of topics which can be covered in roughly two hours.
- Present each definition in full. For each result, present the statement in full, and include a reference for its proof. If possible, include a proof sketch, or at least some main ideas. If most of your talk is a few big proofs, add more details on each proof, perhaps including most of each proof but skipping the calculations.

Aim for about 2–3 pages.

To help you, I have attached an example in the following pages.

Expanders

Sources:

- Hoory, Linial, and Wigderson: Expander graphs and their applications.
- Lecture notes of Ola Svensson (`http://theory.epfl.ch/courses/topicstcs/Lecture3.pdf`).

Expanders are a special type of graphs (or rather, as we explain below, several different types of graphs) that are very useful in theoretical computer science. In some sense, they are a quasirandom object that capture many of the properties of random graphs. Expanders are the most useful in theoretical computer science when they have bounded degree, and so in this lecture all our expanders will be d-regular graphs for some constant d.

Expanders Expanders come in two main varieties: expanders and bipartite expanders. A graph G on n vertices is an expander if every set S of at most $n/2$ vertices is connected to its complement by at least $h|S|$ edges, where $h > 0$ is a constant. As can be seen, the definition of expanders doesn't say much for a single graph (other than it being connected), and is properly a property of a *sequence* of graphs: a sequence of graphs $G_1, G_2, \ldots$ is an expander sequence if they are all expanders using the preceding definition, with a *common* h.

Expanders also have a spectral definition. The adjacency matrix of a d-regular graph always has d as the eigenvalue of largest magnitude, the constant vectors being eigenvectors. We denote by λ_2 the second largest eigenvalue, and call $d - \lambda_2$ the *spectral gap.* A graph is a *spectral expander* if $d - \lambda_2$ is bounded from below by a constant (again, this really defined when a sequence of graphs is a sequence of spectral expanders). It turns out that the two definitions are equivalent, as shown by the Cheeger inequality:[1]

$$\frac{d - \lambda_2}{2} \leq h \leq \sqrt{2d(d - \lambda_2)},$$

where h is the best constant that can be used in the definition above. We will only prove the first inequality, since the proof of the second one is a bit involved. The proof of the first part $h \geq \frac{d-\lambda_2}{2}$ involves decomposing the characteristic vector of a set S into a constant component and its orthogonal complement, and then follows using a simple spectral computation. *Copy the easy part of the proof of Theorem 4.11 in Hoory et al.*

[1]Note: Hoory et al. use λ_2 in the introduction, but later on $\lambda = \max(\lambda_2, |\lambda_n|)$, where λ_n is the minimal eigenvalue. Which is correct?

Alon and Boppana showed that $\lambda_2 \geq 2\sqrt{d-1} - o(1)$ (where $o(1) \to 0$ as $n \to \infty$); here $2\sqrt{d-1}$ is the second eigenvalue of the *infinite* d-ary tree.

Bipartite expanders In many applications it is more natural to consider the bipartite analog: there are two sets L, R of n vertices each, and the degree of each vertex in L is d (we don't care about the degrees of vertices in R). Such a graph is an (α, γ)-expander if every set $S \subseteq L$ containing at most αn vertices has at least $\gamma|S|$ neighbors. A random bipartite graph of this form is an (α, γ)-expander with high probability, for appropriate values of α and γ. The proof uses a straightforward first moment calculation, estimating the sum

$$\sum_{s \leq \alpha n} \binom{n}{s} \binom{n}{\gamma s} \left(\frac{\gamma s}{n}\right)^{ds},$$

where $(\gamma s/n)^{ds}$ is the probability that all ds edges emanating from a specific set of size s point at a specific set of size γs. *Copy the proof in Ola Svensson's lecture notes.*

Properties of expanders One of the most important properties of expanders is the expander mixing lemma. The lemma states that in a d-regular graph with $\lambda = \max(\lambda_2, |\lambda_n|)$, the number of edges between any two sets S, T deviates from the "expected" number $\frac{d}{n}|S||T|$ by at most $\lambda\sqrt{|S||T|}$. This can be used to show that expanders have diameter $O(\log n)$, and to show that the maximum independent set has size at most $\frac{\lambda}{d}n$. The proof is very similar to the proof of $\frac{d-\lambda_2}{2} \leq h$, using Cauchy–Schwartz at one point instead of a simpler bound (this explains why we have to use λ rather than just λ_2). *Copy the proof of Lemma 2.5 of Hoory et al. and the applications thereafter.*

Explicit constructions of expanders *Section 2.2 of Hoory et al. contains some simple examples. The proofs are too difficult to give here.* (In your assignment, it is better to copy these examples since they are relatively simple.)

Applications of expanders *Choose one of the examples in Section 1 of Hoory et al.* (In your assignment, please be more specific in such cases!)