1. **Convergence to Poisson distribution.** If \( p = \frac{\log n + c}{n} \), then the expected number of isolated vertices tends to \( \lambda = e^{-c} \). Let \( X \) be the number of isolated vertices. In class we showed that \( \Pr[X = 0] \to e^{-\lambda} \). Show that for any integer \( k \geq 0 \),
\[
\Pr[X = k] \to e^{-\lambda} \frac{\lambda^k}{k!}.
\]
In other words, \( X \) converges in distribution to a Poisson distribution with expectation \( \lambda \).

2. **Convergence to non-Poisson distribution.** Let \( H \) be a strictly balanced graph, and let \( H^{(2)} \) consist of two disjoint copies of \( H \) (on twice as many vertices). Determine the asymptotic distribution of the number of copies of \( H^{(2)} \) in \( G(n,p) \) for \( p = cn^{-1/d(H)} \) (recall that \( d(H) \) is half the average degree of \( H \)). (You may find it easier to start with the special case in which \( H \) is a single edge.)

3. **Poisson multivariate distribution.** If \( p = c/n \) then the distribution of the degree of a single vertex tends to a Poisson distribution with expectation \( c \). Show that for any constant \( t \), the joint distribution of the degrees of \( t \) vertices tends to the distribution of \( t \) independent copies of the same Poisson distribution.

4. **Normal distribution.** If \( p = c/n \) then the probability that a single vertex is isolated tends to \( e^{-c} \). Consider the following random variables:
   - \( X \) is the number of isolated vertices in \( G(n,p) \).
   - \( X' \sim \text{Bin}(n, e^{-c}) \).
   - \( Y = \frac{X - e^{-c}n}{\sqrt{e^{-c}(1-e^{-c})n}} \).
   - \( Y' = \frac{X' - e^{-c}n}{\sqrt{e^{-c}(1-e^{-c})n}} \).

   It is known that for each \( k \), \( \mathbb{E}[Y^k] \to \mathbb{E}[N(0,1)^k] \), where \( N(0,1) \) is the standard normal distribution (see for example Section 8 of this); when \( k \) is odd, \( \mathbb{E}[N(0,1)^k] = 0 \), and when \( k \) is even, \( \mathbb{E}[N(0,1)^k] = \frac{k!}{2^{k/2}(k/2)!} \).

   (a) Show that for each \( k \), the expected number of \( k \)-tuples of (distinct) isolated vertices is asymptotic to \( e^{-ck}n^k \).

   (b) Deduce that \( \mathbb{E}[Y^k] \to \mathbb{E}[N(0,1)^k] \) for each \( k \) by comparing \( Y \) to \( Y' \) (Wrong!)

   (c) It is known that part (b) implies that \( Y \) converges in distribution to \( N(0,1) \), that is, for each \( t \), \( \Pr[Y < t] \to \Pr[N(0,1) < t] \). What does \( X \) converge to in distribution?