Random Graphs: Assignment 2

Yuval Filmus

Due January 8, 2017

1. **Convergence to Poisson distribution.** If $p = \frac{\log n + c}{n}$, then the expected number of isolated vertices tends to $\lambda = e^{-c}$. Let X be the number of isolated vertices. In class we showed that $\Pr[X = 0] \to e^{-\lambda}$. Show that for any integer $k \geq 0$,
$$\Pr[X = k] \to e^{-\lambda} \frac{\lambda^k}{k!}.$$
In other words, X converges in distribution to a Poisson distribution with expectation λ.

2. **Convergence to non-Poisson distribution.** Let H be a strictly balanced graph, and let $H^{(2)}$ consist of two disjoint copies of H (on twice as many vertices). Determine the asymptotic distribution of the number of copies of $H^{(2)}$ in $G(n,p)$ for $p = cn^{-1/d(H)}$ (recall that $d(H)$ is half the average degree of H). (You may find it easier to start with the special case in which H is a single edge.)

3. **Poisson multivariate distribution.** If $p = c/n$ then the distribution of the degree of a single vertex tends to a Poisson distribution with expectation c. Show that for any constant t, the joint distribution of the degrees of t vertices tends to the distribution of t independent copies of the same Poisson distribution.

4. **Normal distribution.** If $p = c/n$ then the probability that a single vertex is isolated tends to e^{-c}. Consider the following random variables:
   - X is the number of isolated vertices in $G(n,p)$.
   - $X' \sim \mathrm{Bin}(n, e^{-c})$.
   - $Y = \frac{X - e^{-c}n}{\sqrt{e^{-c}(1-e^{-c})n}}$.
   - $Y' = \frac{X' - e^{-c}n}{\sqrt{e^{-c}(1-e^{-c})n}}$.

   It is known that for each k, $\mathbb{E}[Y'^k] \to \mathbb{E}[N(0,1)^k]$, where $N(0,1)$ is the standard normal distribution (see for example Section 8 of this); when k is odd, $\mathbb{E}[N(0,1)^k] = 0$, and when k is even, $\mathbb{E}[N(0,1)^k] = \frac{k!}{2^{k/2}(k/2)!}$.

   (a) Show that for each k, the expected number of k-tuples of (distinct) isolated vertices is asymptotic to $e^{-kc} n^{\underline{k}}$.

   (b) Deduce that $\mathbb{E}[Y^k] \to \mathbb{E}[N(0,1)^k]$ for each k by comparing Y to Y'. (Wrong!)

   (c) It is known that part (b) implies that Y converges in distribution to $N(0,1)$, that is, for each t, $\Pr[Y < t] \to \Pr[N(0,1) < t]$. What does X converge to in distribution?