1. Random countable graphs. A countable graph is a graph whose vertex set is the natural numbers. A countable graph is saturated if for every two disjoint finite sets of vertices $A, B$ there exists a vertex $v$ such that the graph contains all edges $(v, a)$ for $a \in A$ and none of the edges $(v, b)$ for $b \in B$. We saw in class that any two saturated graphs are isomorphic.

In class we considered the $G(\omega, p)$ model, in which each edge is in the graph with probability $p$. We showed that when $p \in (0, 1)$, $G(\omega, p)$ is almost surely saturated. In this exercise we consider a generalized model $G(\omega, f)$, where $f : \mathbb{N} \to [0, 1]$. In this model, the edge $(i, j)$ belongs to the graph with probability $f(\max(i, j))$.

(a) Show that if $-\log \min(f, 1-f) = o(\log n)$ then $G(\omega, f)$ is almost surely saturated.

(b) Show that if $-\log \min(f, 1-f) = \Omega(\log n)$ then $G(\omega, f)$ is almost surely not saturated.

2. Unicyclic components. Show that if $p = \omega(1/n)$ then with high probability $G(n, p)$ contains no connected components with exactly one cycle.

3. Bicyclic components. We saw in class that if $p = 1/n - \omega(1/n^{4/3})$ then with high probability all connected components of $G(n, p)$ contain at most one cycle. It is tempting to conjecture that if $p = 1/n - o(1/n^{4/3})$ then with high probability some connected component of $G(n, p)$ contains more than one cycle.

Experimentally determine the probability that all connected components of $G(n, 1/n)$ or of $G(n, n/2)$ (your choice) contain at most one cycle in the limit $n \to \infty$. Describe your experiments, their results, and a conjectured value for the limit.

4. Diameter. Show that if $p = c/n$ for $c < 1$ and $f(n) = \omega(1)$ then with high probability, the largest diameter $D$ of a connected component of $G(n, p)$ satisfies

$$\left| D - \frac{\log n}{\log(1/c)} \right| < f(n).$$

5. Edges in giant component. Show that if $p = c/n$ for $c > 1$ then with high probability the giant component of $G(n, p)$ contains $\left(1 - x^2/c^2\right)(c/2)n \pm o(n)$ edges, where $x$ is the unique solution to $xe^{1-x} = ce^{1-c}$ in $(0, 1)$.

---

1 Exercise 2.4.2 in the textbook.
2 Exercise 2.4.6 in the textbook.
3 Cf. Theorem 2.14 in the textbook.