Random Graphs: Assignment 1

Yuval Filmus

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1. **Random countable graphs.** A *countable graph* is a graph whose vertex set is the natural numbers. A countable graph is *saturated* if for every two disjoint finite sets of vertices A, B there exists a vertex v such that the graph contains all edges (v, a) for $a \in A$ and none of the edges (v, b) for $b \in B$. We saw in class that any two saturated graphs are isomorphic.

   In class we considered the $G(\omega, p)$ model, in which each edge is in the graph with probability p. We showed that when $p \in (0, 1)$, $G(\omega, p)$ is almost surely saturated. In this exercise we consider a generalized model $G(\omega, f)$, where $f : \mathbb{N} \to [0, 1]$. In this model, the edge (i, j) belongs to the graph with probability $f(\max(i, j))$.

   (a) Show that if $-\log \min(f, 1 - f) = o(\log n)$ then $G(\omega, f)$ is almost surely saturated.

   (b) Show that if $-\log \min(f, 1 - f) = \Omega(\log n)$ then $G(\omega, f)$ is almost surely not saturated.

2. **Unicyclic components**[1]**.** Show that if $p = \omega(1/n)$ then with high probability $G(n, p)$ contains no connected components with *exactly* one cycle.

3. **Bicyclic components.** We saw in class that if $p = 1/n - \omega(1/n^{4/3})$ then with high probability all connected components of $G(n, p)$ contain at most one cycle. It is tempting to conjecture that if $p = 1/n - o(1/n^{4/3})$ then with high probability some connected component of $G(n, p)$ contains more than one cycle.

   Experimentally determine the probability that all connected components of $G(n, 1/n)$ or of $G(n, n/2)$ (your choice) contain at most one cycle in the limit $n \to \infty$. Describe your experiments, their results, and a conjectured value for the limit.

4. **Diameter**[2]**.** Show that if $p = c/n$ for $c < 1$ and $f(n) = \omega(1)$ then with high probability, the largest diameter D of a connected component of $G(n, p)$ satisfies

$$\left| D - \frac{\log n}{\log(1/c)} \right| < f(n).$$

5. **Edges in giant component**[3]**.** Show that if $p = c/n$ for $c > 1$ then with high probability the giant component of $G(n, p)$ contains $(1 - x^2/c^2)(c/2)n \pm o(n)$ edges, where x is the unique solution to $xe^{1-x} = ce^{1-c}$ in $(0, 1)$.

[1] Exercise 2.4.2 in the textbook.

[2] Exercise 2.4.6 in the textbook.

[3] Cf. Theorem 2.14 in the textbook.