Assignment 3
Technion 236646
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Question 1 Let $G = (U, V, E)$ be a bipartite graph satisfying the following properties, for some constants $d$ and $\alpha > 0$:

(i) $|U| = n + 1$ and $|V| = n$.
(ii) Every vertex in $U$ has degree $d$.
(iii) If $S \subseteq U$ has size at most $\alpha n$ then there are at least $|S|$ vertices in $V$ which are neighbors of exactly one vertex in $S$.

The graph pigeonhole principle corresponding to $G$ has variables $x_{uv}$ for all $(u, v) \in E$, and the following axioms:

- Pigeon axioms: For every $u \in U$, the axiom $\bigvee_v (u, v) \in E x_{uv}$.
- Hole axioms: For every $u_1, u_2 \in U$ and $v \in V$ such that $(u_1, v), (u_2, v) \in E$ and $u_1 \neq u_2$, the following axiom: $x_{u_1v} \vee x_{u_2v}$.

(a) Show that the graph pigeonhole principle is unsatisfiable.
(b) For a clause $C$, let $w(C)$ be the minimal number of pigeon axioms which, together with all hole axioms, logically imply $C$. Show that $w(\bot) > \alpha n$. Hint: use Hall’s theorem.
(c) Show that if $w(C) \leq \alpha n$ then $C$ has width at least $w(C)$.
(d) Deduce that the width of refuting the graph pigeonhole principle is $\Omega(\alpha n)$.
(e) Conclude that the length required to refute the graph pigeonhole principle is $2^{\Omega(\alpha^2 n/d)}$.
(f) It is known that there exist constant $d, \alpha$ such that a random subgraph of $K_{n+1,n}$ (the complete bipartite graph on $(n + 1) + n$ vertices) satisfying property (ii) also satisfies property (iii) with probability $1 - o(1)$.

The pigeonhole principle is the graph pigeonhole principle corresponding to $K_{n+1,n}$.

Show that the length required to refute the pigeonhole principle is $2^{\Omega(n)}$. 

Question 2  Tarsi showed that if $S$ is a minimally unsatisfiable set of clauses (meaning that $S$ is unsatisfiable, but any proper subset is satisfiable) then $S$ mentions fewer than $|S|$ variables.

(a) Suppose that $S$ is an unsatisfiable set of clauses. Show that $S$ can be refuted in Resolution in length $O(2^{|S|})$.

(b) Suppose that $S$ logically implies a clause $C$. Show that $C$ can be derived from $S$ in Resolution (including the weakening rule) in length $O(2^{|S|})$.

(c) Semantic $d$-ary Resolution is the following proof system. Each line is a clause, which is either an axiom, or logically follows from up to $d$ earlier lines. (Resolution is essentially the case $d = 2$, if we allow the “weakening cut” rule $C \lor x, D \lor \bar{x} \vdash C \lor D \lor E$.)

Show that if a CNF can be refuted in Semantic $d$-ary Resolution using $\ell$ lines, then it can also be refuted in Resolution using $O(2^d \ell)$ lines.

(d) Show that for every constant $d$, we can check the validity of a Semantic $d$-ary Resolution proof in time $2^{O(d) n^{O(1)}}$.

Question 3  Consider a random walk process on the line $\{0, \ldots, n\}$ satisfying the following two properties:

1. If at time $t$ the walk is at $n$, then at time $t + 1$ it is at $n - 1$.

2. If at time $t$ the walk is at $0 < d < n$, then at time $t + 1$ it is either at $d - 1$ or at $d + 1$; and furthermore, it is at $d - 1$ with probability at least $1/2$.

It is known that such a process hits 0 after $O(n^2)$ steps, in expectation. Use this to analyze the performance of Schöning’s algorithm on 2CNFs.