Boolean Function Analysis — Assignment 3

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Let S_n denote the *symmetric group* of all permutations of $[n] = \{1, \ldots, n\}$. We represent a permutation by the corresponding $n \times n$ permutation matrix $(x_{ij})_{i,j \in [n]}$. Such a matrix describes a permutation if the following properties hold:

(i) Each entry equals 0 or 1: $x_{ij} \in \{0, 1\}$ for all $i, j \in [n]$.

(ii) Each row contains a unique 1: for all $i \in [n]$, $\sum_j x_{ij} = 1$.

(iii) Each columns contains a unique 1: for all $j \in [n]$, $\sum_i x_{ij} = 1$.

A function on S_n has degree 1 (by which we really mean degree *at most* 1) if it can be written in the form

$$\sum_{ij} c_{ij} x_{ij}.$$

1. Another natural definition for degree 1 functions defines them as all functions that can be written in the form

   $$c + \sum_{ij} c_{ij} x_{ij}.$$

   Show that the two definitions coincide.

   This follows from $c = c(x_{11} + \cdots + x_{1n})$.

2. The space of all degree 1 functions is a subspace of all real-valued functions on the symmetric group. In this question, we will determine its dimension.

   (a) Let f be a degree 1 function. With each representation

   $$f = c + \sum_{ij} c_{ij} x_{ij}$$

   we associate a potential value $\sum_{ij} c_{ij}^2$. Standard topology shows that some representation minimizes the potential value.
   Show that if a representation minimizes the potential value then it satisfies the following constraints:

   (i) For all $i \in [n]$, $\sum_j c_{ij} = 0$.
   (ii) For all $j \in [n]$, $\sum_i c_{ij} = 0$.

We call such a representation *canonical*.

Suppose that one of the constraints is violated, say $c_i := \sum_j c_{ij} \neq 0$ for some i. Let $d = c + c_i/n$, $d_{i'j} = c_{ij}$ if $i' \neq i$, and $d_{ij} = c_{ij} - c_i/n$. Then

$$f = d + \sum_{ij} d_{ij} x_{ij}$$

is another representation of f, and we have

$$\sum_{ij} c_{ij}^2 - \sum_{ij} d_{ij}^2 = \sum_j (c_{ij}^2 - (c_{ij} - c_i/n)^2) = \frac{2c_i}{n} \sum_j c_{ij} - n\frac{c_i^2}{n^2} = \frac{c_i^2}{n} > 0,$$

showing that the original representation did not minimize the potential value.

(b) Consider a function f represented in canonical form. Calculate the expectation of f over all permutations, and the expectation of f over all permutations mapping i to j.

The expectation of f is

$$c + \sum_{ij} c_{ij} \frac{1}{n} = c.$$

The expectation of f over all permutations mapping i to j is

$$c + c_{ij} + \sum_{\substack{i' \neq i \\ j' \neq j}} c_{i'j'} \frac{1}{n-1}.$$

For every fixed $i' \neq i$, $\sum_{j' \neq j} c_{i'j'} = -c_{i'j}$. Summing over all $i' \neq i$ gives c_{ij}. Hence the expectation is

$$c + \frac{n}{n-1} c_{ij}.$$

(c) Conclude the dimension of the space of all degree 1 functions.

If f evaluates to zero over the entire symmetric group, then the preceding item shows that any canonical representation of f satisfies $c = 0$ and $c + \frac{n}{n-1} c_{ij} = 0$ for all $i, j \in [n]$, implying that $c_{ij} = 0$ for all $i, j \in [n]$. This implies that every function f has a *unique* canonical representation: if f, g are two such representations, $f - g$ evaluates to zero over the entire symmetric group, and so their coefficients coincide.

The canonical representation is determined by the coefficients c and c_{ij} for $i, j < n$. In contrast, every such choice of coefficients corresponds to the canonical representation of some function: we can define $c_{in} = -c_{i1} - \cdots - c_{i(n-1)}$ for $i < n$ and $c_{nj} = -c_{1j} - \cdots - c_{(n-1)j}$ for $j \in [n]$. The remaining constraint is automatically satisfied since

$$\sum_{j=1}^{n} c_{nj} = -\sum_{j=1}^{n} \sum_{i=1}^{n-1} c_{ij} = -\sum_{i=1}^{n-1} \sum_{j=1}^{n} c_{ij} = 0.$$

We conclude that the space has dimension $(n-1)^2 + 1$.

3. In this question, we will determine all Boolean degree 1 functions.
   Let $f = \sum_{ij} c_{ij} x_{ij}$ be Boolean: $f(x) \in \{0, 1\}$ for every $x \in S_n$.

   (a) An $n \times n$ matrix $(x_{ij})_{i,j\in[n]}$ is *bistochastic* if the following properties hold:

      (i) $x_{ij} \geq 0$ for all $i, j \in [n]$.

      (ii) For all $i \in [n]$, $\sum_j x_{ij} = 1$.

      (iii) For all $j \in [n]$, $\sum_i x_{ij} = 1$.

      Birkhoff and von Neumann showed that every bistochastic matrix can be written as a convex combination of permutation matrices.[1]

      Show that $0 \leq f(x) \leq 1$ for all bistochastic x.

      According to the Birkhoff–von Neumann theorem, $x = c_1 y_1 + \cdots + c_m y_m$, where $c_1, \ldots, c_m \geq 0$ and $c_1 + \cdots + c_m = 1$. Since f is linear, $f(x) = c_1 f(y_1) + \cdots + c_m f(y_m)$. Since $f(y_1), \ldots, f(y_m) \in \{0, 1\}$, we see that $0 \leq f(x) \leq 1$.

   (b) The set of all bistochastic matrices forms a polytope known as the *Birkhoff polytope*. A *face* of the polytope consists of all bistochastic matrices satisfying $x_{i_1 j_1} = \cdots = x_{i_m j_m} = 0$, where $0 \leq m \leq n^2$. If ℓ is any linear function in the x_{ij}, then the subset of the Birkhoff polytope on which ℓ attains its minimum value is a face of the polytope.

      Show that if f is not constant (as a function on the symmetric group), then there are $i, j \in [n]$ such that if $x_{ij} = 1$ then $f(x) = 1$.

      Since $f \not\equiv 1$, the minimum of f on the Birkhoff polytope is 0. This minimum is attained on some face $x_{i_1 j_1} = \cdots = x_{i_m j_m} = 0$, and so $f(x) = 0$ iff $x_{i_1 j_1} = \cdots = x_{i_m j_m} = 0$. Since $f \not\equiv 0$, the face cannot consist of the entire polytope, and so $m \geq 1$. Consequently, if $f(x) = 0$ then $x_{i_1 j_1} = 0$. This implies that if $x_{i_1 j_1} = 1$ then $f(x) \neq 0$. Since f is Boolean, if $x_{i_1 j_1} = 1$ then $f(x) = 1$.

   (c) Show that f can be written as a sum $f = x_{i_1 j_1} + \cdots + x_{i_m j_m}$, where $0 \leq m \leq n$.

      If $f \equiv 1$ then $f = x_{11} + \cdots + x_{1n}$, so we can assume that this is not the case. Define a sequence $f_0 = f, \ldots, f_m = 0$ of Boolean degree 1 functions as follows. If $f_k \neq 0$, then apply the previous item to find $i_{k+1}, j_{k+1} \in [n]$ such that $f_k(x) = 1$ whenever $x_{i_{k+1} j_{k+1}} = 1$. It follows that $f_{k+1} = f_k - x_{i_{k+1} j_{k+1}}$ is also a Boolean degree 1 function. The process must terminate in at most n steps since $\mathbb{E}[f_{k+1}] = \mathbb{E}[f_k] - 1/n$. Since $f_m = 0$, we obtain the required representation of f.

   (d) Give an explicit description of all Boolean degree 1 functions on the symmetric group.

      The previous item shows that $f = x_{i_1 j_1} + \cdots + x_{i_m j_m}$. Since f is Boolean, we cannot have $x_{i_s j_s} = x_{i_t j_t} = 1$ for $s \neq t$. This means that every two entries must be on the same row or on the same column. If $x_{i_r j_r}$ is on the same row as $x_{i_s j_s}$ and on the same column as $x_{i_t j_t}$ then $x_{i_s j_s}, x_{i_t j_t}$ are not on the same row or column. Consequently, either all of $x_{i_1 j_1}, \ldots, x_{i_m j_m}$ are on the same row, or all of them are on the same column. Therefore f is either of the form

      $$f = \sum_{j \in J} x_{ij}$$

[1] A matrix x is a convex combination of matrices $y_1, \ldots, y_m$ if there exist $c_1, \ldots, c_m \geq 0$ summing to 1 such that $x = c_1 y_1 + \cdots + c_m y_m$.

for some $i \in [n]$ and $J \subseteq [n]$, or of the form

$$f = \sum_{i \in I} x_{ij}$$

for some $j \in [n]$ and $I \subseteq [n]$.

4. The alternating group A_n is the subgroup of the symmetric group S_n consisting of all even permutations. When $n = 4$, A_n consists of the following 12 permutations:

   (i) The identity.

   (ii) Products of two disjoint 2-cycles: $(12)(34), (13)(24), (14)(23)$.

   (iii) 3-cycles: $(123), (132), (124), (142), (134), (143), (234), (243)$.

   Consider the function

$$f = \frac{x_{11} + x_{23} - x_{32} + x_{44}}{2}.$$

   (a) Show that as a function on A_4, f is Boolean.
   The function f is the indicator of the set $e, (123), (234)$.

   (b) Show that as a function on S_4, f is not Boolean.
   The value of f on (12) is $1/2$.

   (c) Conclude that the characterization of Boolean degree 1 functions on the symmetric group fails for the alternating group, at least when $n = 4$.
   If f could be written in the form $\sum_{j \in J} x_{ij}$ or $\sum_{i \in I} x_{ij}$, then f would have been Boolean as a function on S_4.