Boolean Function Analysis — Assignment 1

Yuval Filmus

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1. Let $f, g\colon \{\pm 1\}^n \to \mathbb{R}$ and $c \in \mathbb{R}$. Express the Fourier expansions of $cf, f+g, fg$ in terms of the Fourier expansions of f and g. That is, for $h \in \{cf, f+g, fg\}$ and $S \subseteq [n]$, give a formula for $\hat{h}(S)$ in terms of the Fourier coefficients of f and g.

   We have
   $$cf(x) = c \sum_{S \subseteq [n]} \hat{f}(S) x_S = \sum_{S \subseteq [n]} c\hat{f}(S) x_S,$$
   and so $\widehat{cf}(S) = c\hat{f}(S)$.

   Similarly,
   $$f(x) + g(x) = \sum_{S \subseteq [n]} \hat{f}(S) + \sum_{S \subseteq [n]} \hat{g}(S) = \sum_{S \subseteq [n]} (\hat{f}(S) + \hat{g}(S)) x_S,$$
   and so $\widehat{f+g}(S) = \hat{f}(S)\hat{g}(S)$.

   Finally,
   $$f(x)g(x) = \left(\sum_{S \subseteq [n]} \hat{f}(S) x_S\right)\left(\sum_{T \subseteq [n]} \hat{g}(T) x_T\right) = \sum_{S,T \subseteq [n]} \hat{f}(S)\hat{g}(T) x_S x_T.$$
   Since $x_S x_T = x_{S \Delta T}$, we can read off
   $$\widehat{fg}(R) = \sum_{S \subseteq [n]} \hat{f}(S)\hat{g}(S \Delta R),$$
   since $S \Delta T = R$ iff $T = S \Delta R$.

2. The majority function on $n = 2m+1$ inputs is a function from $\{\pm 1\}^n$ to $\{\pm 1\}$ given by the formula
   $$\mathrm{Maj}_n(x_1, \ldots, x_n) = \mathrm{sgn}(x_1 + \cdots + x_n).$$
   Determine the Fourier expansions of $\mathrm{Maj}_1, \mathrm{Maj}_3, \mathrm{Maj}_5$ (we have seen one of these in class).

   When $n = 1$, Maj_1 is the identity, and so its Fourier expansion is $\mathrm{Maj}_1(x_1) = x_1$. The case $n = 3$ we have seen in class:
   $$\mathrm{Maj}_3(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3 - x_1x_2x_3}{2}.$$
   When $n = 5$, we know that Maj_5 is an odd function, and so due to symmetry,
   $$\mathrm{Maj}_5(x_1, x_2, x_3, x_4, x_5) = A\sum_{i=1}^{5} x_i + B \sum_{1 \le i < j < k \le 5} x_i x_j x_k + C x_1 x_2 x_3 x_4 x_5.$$
   Considering the inputs $1,1,1,1,1$; $1,1,1,1-1$; $1,1,1,-1,-1$, we get a system of linear equations:
   $$5A + 10B + C = 1$$
   $$3A - 2B - C = 1$$
   $$A - 2B + C = 1$$
   The solution is $A = 3/8$, $B = -1/8$, $C = 3/8$.

3. In this question we outline an alternative argument for classifying all polymorphisms of Maj_3. Suppose that $f\colon \{\pm 1\}^n \to \{\pm 1\}$ satisfies

$$f(\mathrm{Maj}_3(x_1, y_1, z_1), \ldots, \mathrm{Maj}_3(x_n, y_n, z_n)) = \mathrm{Maj}_3(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), f(z_1, \ldots, z_n))$$

for all $x, y, z \in \{\pm 1\}^n$.

(a) What is the Fourier expansion of $f(\mathrm{Maj}_3(x_1, y_1, z_1), \ldots, \mathrm{Maj}_3(x_n, y_n, z_n))$?
Using the Fourier expansion of Maj_3, we get

$$\sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} \frac{x_i + y_i + z_i - x_i y_i z_i}{2} = \sum_{A,B,C,D \text{ disjoint}} \hat{f}(A \cup B \cup C \cup D) 2^{-|A|+|B|+|C|+|D|} (-1)^{|D|} x_{A \cup D} y_{B \cup D} z_{C \cup D}.$$

(b) What is the Fourier expansion of $\mathrm{Maj}_3(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), f(z_1, \ldots, z_n))$?
Again using the Fourier expansion of Maj_3, we get

$$\frac{f(x) + f(y) + f(z) - f(x)f(y)f(z)}{2} = \frac{1}{2} \sum_{S \subseteq [n]} \hat{f}(S) x_S + \frac{1}{2} \sum_{T \subseteq [n]} \hat{f}(T) y_T + \frac{1}{2} \sum_{U \subseteq [n]} \hat{f}(U) z_U - \frac{1}{2} \sum_{S,T,U \subseteq [n]} \hat{f}(S)\hat{f}(T)\hat{f}(U) x_S y_T z_U,$$

which corresponds to the Fourier expansion

$$\frac{1}{2}\hat{f}(\emptyset)(3 - \hat{f}(\emptyset)^2) + \frac{1}{2} \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} \hat{f}(S)(1 - \hat{f}(\emptyset)^2) x_S + \frac{1}{2} \sum_{\substack{T \subseteq [n] \\ T \neq \emptyset}} \hat{f}(T)(1 - \hat{f}(\emptyset)^2) y_T + \frac{1}{2} \sum_{\substack{U \subseteq [n] \\ U \neq \emptyset}} \hat{f}(U)(1 - \hat{f}(\emptyset)^2) z_U - \frac{1}{2} \sum_{\substack{S,T,U \subseteq [n] \\ \text{at least two of} \\ S,T,U \text{ non-empty}}} \hat{f}(S)\hat{f}(T)\hat{f}(U) x_S y_T z_U.$$

(c) Compare the constant coefficients (the coefficients of the empty monomial 1) to deduce that $\mathbb{E}[f] \in \{0, \pm 1\}$, and so either $f = \pm 1$ or $\mathbb{E}[f] = 0$ (this is identical to what we did in class).
The empty coefficient in $f(\mathrm{Maj}_3(x, y, z))$ corresponds to $A = B = C = D = \emptyset$, and so it is $\hat{f}(\emptyset)$. The empty coefficient in $\mathrm{Maj}_3(f(x), f(y), f(z))$ is $\frac{1}{2}\hat{f}(\emptyset)(3 - \hat{f}(\emptyset)^2)$. Equating both sides, we conclude that $\hat{f}(\emptyset)(1 - \hat{f}(\emptyset)^2) = 0$, and so $\hat{f}(\emptyset) \in \{0, \pm 1\}$. If $\hat{f}(\emptyset) = \pm 1$ then $\mathbb{E}[f] = \pm 1$ and so $f = \pm 1$, and if $\hat{f}(\emptyset) = 0$ then $\mathbb{E}[f] = 0$.

(d) Assume that $\mathbb{E}[f] = 0$, and let $S \subseteq [n]$ be non-empty. Compare the coefficients of x_S in both Fourier expansions to conclude that either $|S| = 1$ or $\hat{f}(S) = 0$.
The coefficient of x_S in $f(\mathrm{Maj}_3(x, y, z))$ corresponds to $A = S$ and $B = C = D = \emptyset$, and so it is $2^{-|S|}\hat{f}(S)$. The coefficient of x_S in $\mathrm{Maj}_3(f(x), f(y), f(z))$ is $\frac{1}{2}\hat{f}(S)(1 - \hat{f}(\emptyset)^2) = \frac{1}{2}\hat{f}(S)$. Thus $(2^{-|S|} - 1/2)\hat{f}(S) = 0$, and so either $|S| = 1$ or $\hat{f}(S) = 0$.

(e) Conclude that either $f = \pm 1$ or $f = \pm x_i$ (this is identical to what we did in class).
Part (c) shows that either $f = \pm 1$ or $\mathbb{E}[f] = 0$. In the latter case, part (d) shows that $\deg(f) = 1$, and so either $f = \pm 1$ (which is impossible since $\mathbb{E}[f] = 0$) or $f = \pm x_i$ for some $i \in [n]$.

4. In this question you will extend the analysis of linearity testing from the test $f(xy) = f(x)f(y)$ to the test $f(xyz) = f(x)f(y)f(z)$.

   (a) Express $\Pr_{x,y,z}[f(xyz) = f(x)f(y)f(z)]$ in terms of the Fourier coefficients of f.
   As seen in class,

$$\Pr[f(xyz) = f(x)f(y)f(z)] = \frac{1}{2} + \frac{1}{2}\mathbb{E}[f(xyz)f(x)f(y)f(z)].$$

   The expectation equals

$$\sum_{R,S,T,U\subseteq[n]} \hat{f}(R)\hat{f}(S)\hat{f}(T)\hat{f}(U)\mathbb{E}[(xyz)_R x_S y_T z_U] = \sum_{R,S,T,U\subseteq[n]} \hat{f}(R)\hat{f}(S)\hat{f}(T)\hat{f}(U)\mathbb{E}[x_{R\Delta S} y_{R\Delta T} z_{R\Delta U}] = \sum_{R\subseteq[n]} \hat{f}(R)^4.$$

   Therefore

$$\Pr[f(xyz) = f(x)f(y)f(z)] = \frac{1}{2} + \frac{1}{2}\sum_{S\subseteq[n]} \hat{f}(S)^4.$$

   (b) Determine all functions $f\colon \{\pm1\}^n \to \{\pm1\}$ satisfying $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in \{\pm1\}^n$.
   If $f(xyz) = f(x)f(y)f(z)$ then $\Pr[f(xyz) = f(x)f(y)f(z)] = 1$, and so using Parseval's identity,

$$1 = \sum_{S\subseteq[n]} \hat{f}(S)^4 \le \sum_{S\subseteq[n]} \hat{f}(S)^2 \cdot \max_{T\subseteq[n]} \hat{f}(T)^2 = \max_{T\subseteq[n]} \hat{f}(T)^2.$$

   Since $\hat{f}(T)^2 \le 1$ for all T, it follows that $\hat{f}(T) = \pm1$ for some T. Invoking Parseval's identity again, we see that $f = \pm x_T$.
   Conversely, if $f = \sigma x_T$, where $\sigma \in \{\pm1\}$, then

$$f(xyz) = \sigma(xyz)_T = \sigma x_T \sigma y_T \sigma z_T = f(x)f(y)f(z),$$

   since $\sigma^3 = \sigma$. We conclude that the solutions are $\pm x_T$ for all $T \subseteq [n]$.

   (c) Show that if $f\colon \{\pm1\}^n \to \{\pm1\}$ satisfies

$$\Pr_{x,y,z}[f(xyz) \ne f(x)f(y)f(z)] \le \epsilon$$

   then $\Pr[f \ne g] = O(\epsilon)$ for some g which satisfies $g(xyz) = g(x)g(y)g(z)$ for all $x, y, z \in \{\pm1\}^n$.
   If $\Pr[f(xyz) = f(x)f(y)f(z)] \ge 1 - \epsilon$ then

$$1 - 2\epsilon \le \sum_{S\subseteq[n]} \hat{f}(S)^4 \le \sum_{S\subseteq[n]} \hat{f}(S)^2 \cdot \max_{T\subseteq[n]} \hat{f}(T)^2 = \max_{T\subseteq[n]} \hat{f}(T)^2.$$

   Therefore there exists some T such that $\hat{f}(T)^2 \ge 1 - 2\epsilon$, implying that $|\hat{f}(T)| \ge 1 - \Omega(\epsilon)$. Let σ be the sign of $\hat{f}(T)$. Then

$$\Pr[f = \sigma x_T] = \frac{1}{2} + \frac{1}{2}\mathbb{E}[f\sigma x_T] = \frac{1}{2} + \frac{1}{2}\sigma\hat{f}(T) \ge \frac{1}{2} + \frac{1}{2}(1 - \Omega(\epsilon)) = 1 - \Omega(\epsilon).$$

   We have shown that $g = \sigma x_T$ satisfies $g(xyz) = g(x)g(y)g(z)$ in part (b), completing the proof.

5. In class we gave a list of all functions $f\colon \{\pm 1\}^n \to \{\pm 1\}$ which have degree at most 1. What are all the functions $f\colon \{\pm 1\}^n \to \{0, \pm 1\}$ which have degree at most 1?

We will show that the solutions are

$$0, \sigma, \sigma x_i, \frac{1+\sigma x_i}{2}, \frac{\sigma x_i + \tau x_j}{2},$$

where $\sigma, \tau \in \{\pm 1\}$ and $i \neq j$.

Let $g(x_0, \ldots, x_n) = x_0 f(x_0 x_1, \ldots, x_0 x_n)$, which also takes values in $\{0, \pm 1\}$. We have

$$g(x_0, \ldots, x_n) = x_0 \left(\hat{f}(\emptyset) + \sum_{i=1}^{n} \hat{f}(\{i\}) x_0 x_i \right) = \sum_{i=0}^{n} c_i x_i,$$

where $c_0 = \hat{f}(\emptyset)$ and $c_i = \hat{f}(\{i\})$ for $i \in [n]$.

The difference of two values in g is always in $\{0, \pm 1, \pm 2\}$. For $S \subseteq [n]$, let δ_S be the input which is -1 on S and 1 on $\overline{S}$. Since

$$\frac{g(\delta_S) - g(\delta_T)}{2} = \sum_{i \in T \setminus S} c_i - \sum_{i \in S \setminus T} c_i,$$

this shows that for all $\sigma_1, \ldots, \sigma_n \in \{0, \pm 1\}$,

$$\sum_{i=0}^{n} \sigma_i c_i \in \{0, \pm\tfrac{1}{2}, \pm 1\}.$$

If $c_i = \pm 1$ for some i, then according to Parseval's identity, $g = \pm x_i$.

Assuming this is not the case, $c_i \in \{0, \frac{1}{2}\}$ for all i. If $c_i, c_j, c_k \neq 0$ for three different indices i, j, k, then by choosing $\sigma_i = 2c_i$, $\sigma_j = 2c_j$, $\sigma_k = 2c_k$ and $\sigma_\ell = 0$ for $\ell \neq i, j, k$, we reach a contradiction, since $\sum_t \sigma_t c_t = \frac{3}{2}$. Therefore at most two c_i's are non-zero.

If exactly one c_i is non-zero then $g = \pm\frac{1}{2} x_i$, which is not $\{0, \pm 1\}$-valued. Hence either $g = 0$ or $g = \sigma \frac{1}{2} x_i + \tau \frac{1}{2} x_j$, for some $\sigma, \tau \in \{\pm 1\}$ and $i \neq j$.

We conclude that g is one of the following:

$$0, \sigma x_i, \frac{\sigma x_i + \tau x_j}{2},$$

where $\sigma, \tau \in \{\pm 1\}$ and $i \neq j$. It is easy to check that all of these are $\{0, \pm 1\}$-valued. Substituting $x_0 = 1$, we get the set of possible f.