Boolean Function Analysis — Assignment 3

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Let S_n denote the *symmetric group* of all permutations of $[n] = \{1, \ldots, n\}$. We represent a permutation by the corresponding $n \times n$ permutation matrix $(x_{ij})_{i,j \in [n]}$. Such a matrix describes a permutation if the following properties hold:

(i) Each entry equals 0 or 1: $x_{ij} \in \{0, 1\}$ for all $i, j \in [n]$.

(ii) Each row contains a unique 1: for all $i \in [n]$, $\sum_j x_{ij} = 1$.

(iii) Each columns contains a unique 1: for all $j \in [n]$, $\sum_i x_{ij} = 1$.

A function on S_n has degree 1 (by which we really mean degree *at most* 1) if it can be written in the form

$$\sum_{ij} c_{ij} x_{ij}.$$

1. Another natural definition for degree 1 functions defines them as all functions that can be written in the form

$$c + \sum_{ij} c_{ij} x_{ij}.$$

   Show that the two definitions coincide.

2. The space of all degree 1 functions is a subspace of all real-valued functions on the symmetric group. In this question, we will determine its dimension.

   (a) Let f be a degree 1 function. With each representation

$$f = c + \sum_{ij} c_{ij} x_{ij}$$

   we associate a potential value $\sum_{ij} c_{ij}^2$. Standard topology shows that some representation minimizes the potential value.
   Show that if a representation minimizes the potential value then it satisfies the following constraints:

   (i) For all $i \in [n]$, $\sum_j c_{ij} = 0$.
   (ii) For all $j \in [n]$, $\sum_i c_{ij} = 0$.

   We call such a representation *canonical*.

   (b) Consider a function f represented in canonical form. Calculate the expectation of f over all permutations, and the expectation of f over all permutations mapping i to j.

   (c) Conclude the dimension of the space of all degree 1 functions.

3. In this question, we will determine all Boolean degree 1 functions.
Let $f = \sum_{ij} c_{ij} x_{ij}$ be Boolean: $f(x) \in \{0, 1\}$ for every $x \in S_n$.

   (a) An $n \times n$ matrix $(x_{ij})_{i,j \in [n]}$ is *bistochastic* if the following properties hold:

      (i) $x_{ij} \geq 0$ for all $i, j \in [n]$.

      (ii) For all $i \in [n]$, $\sum_j x_{ij} = 1$.

      (iii) For all $j \in [n]$, $\sum_i x_{ij} = 1$.

   Birkhoff and von Neumann showed that every bistochastic matrix can be written as a convex combination of permutation matrices.[1]
   Show that $0 \leq f(x) \leq 1$ for all bistochastic x.

   (b) The set of all bistochastic matrices forms a polytope known as the *Birkhoff polytope.* A *face* of the polytope consists of all bistochastic matrices satisfying $x_{i_1 j_1} = \cdots = x_{i_m j_m} = 0$, where $0 \leq m \leq n^2$. If ℓ is any linear function in the x_{ij}, then the subset of the Birkhoff polytope on which ℓ attains its minimum value is a face of the polytope.
   Show that if f is not constant (as a function on the symmetric group), then there are $i, j \in [n]$ such that if $x_{ij} = 1$ then $f(x) = 1$.

   (c) Show that f can be written as a sum $f = x_{i_1 j_1} + \cdots + x_{i_m j_m}$, where $0 \leq m \leq n$.

   (d) Give an explicit description of all Boolean degree 1 functions on the symmetric group.

4. The alternating group A_n is the subgroup of the symmetric group S_n consisting of all even permutations. When $n = 4$, A_n consists of the following 12 permutations:

   (i) The identity.

   (ii) Products of two disjoint 2-cycles: $(12)(34), (13)(24), (14)(23)$.

   (iii) 3-cycles: $(123), (132), (124), (142), (134), (143), (234), (243)$.

   Consider the function
$$f = \frac{x_{11} + x_{23} - x_{32} + x_{44}}{2}.$$

   (a) Show that as a function on A_4, f is Boolean.

   (b) Show that as a function on S_4, f is not Boolean.

   (c) Conclude that the characterization of Boolean degree 1 functions on the symmetric group fails for the alternating group, at least when $n = 4$.

[1] A matrix x is a convex combination of matrices $y_1, \ldots, y_m$ if there exist $c_1, \ldots, c_m \geq 0$ summing to 1 such that $x = c_1 y_1 + \cdots + c_m y_m$.