Boolean Function Analysis — Assignment 2

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1. The proof of the FKN theorem in Section 3 of the lecture notes uses the estimate
$$\mathbb{E}\left[\left(\sum_{i=1}^n c_i x_i\right)^4\right] \leq 3\left(\sum_{i=1}^n c_i^2\right)^2.$$
Show that this follows from hypercontractivity (up to the constant 3).

2. The goal of this exercise is to present an alternative proof of the FKN theorem using the Berry–Esseen theorem, a quantitative version of the central limit theorem.
A special case of the Berry–Esseen theorem states that if $X = \sum_{i=1}^n c_i x_i$, where $\sum_{i=1}^n c_i^2 = 1$ and $|c_i| \leq \delta$ for all i, then for all $t \in \mathbb{R}$,
$$|\Pr[X < t] - \Pr[N(0,1) < t]| \leq \delta,$$
where $N(0,1)$ is the Gaussian distribution with zero mean and unit variance.
   (a) Use the Berry–Esseen to show that the following holds for some constant $c > 0$.
   If $f = \sum_{i=1}^n a_i x_i$ satisfies $1-c \leq \sum_{i=1}^n a_i^2 \leq 1$ and $|a_i| \leq c$ for all i then $\mathbb{E}[\operatorname{dist}(f, \{\pm 1\})^2] \geq c$.
   (b) Explain how to deduce the FKN theorem (in its version for Boolean functions) from this, possibly with a worse error bound.

3. The goal of this exercise is to show that Friedgut's junta theorem fails for bounded functions.
   (a) Let $f(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{\sqrt{n}}$. Calculate $\operatorname{Inf}_1[f]$ and $\operatorname{Inf}[f]$.
   (b) Let $g(x)$ result from clipping $f(x)$ to $[-1,1]$, that is, $g(x) = f(x)$ if $f(x) \in [-1,1]$, $g(x) = -1$ if $f(x) < -1$, and $g(x) = 1$ if $f(x) > 1$. Show that $\operatorname{Inf}[g] = O(1)$.
   (c) Show that for some constants $\epsilon > 0$ and $N \in \mathbb{N}$, if $n \geq N$ and g is ϵ-close to a function $h\colon \{-1,1\}^n \to \mathbb{R}$ (that is, $\mathbb{E}[(g-h)^2] \leq \epsilon$) then h depends on at least $n/2$ variables.

4. The goal of this exercise is to show that the parameters in Friedgut's junta theorem are tight.
   (a) Let $f\colon \{-1,1\}^{2^m+m} \to \{-1,1\}$ be the addressing function $f(x,y) = x_y$ (that is, $x \in \{-1,1\}^{2^m}$, $y \in \{-1,1\}^m$, and we interpret y as an index into x). Calculate the individual influences and the total influence of f.
   (b) Let $g\colon \{-1,1\}^{2^m+m+k} \to \{-1,1\}$ be the function given by $g(x,y,z) = f(x,y)$ if $z = \mathbf{1}$ (where $z \in \{-1,1\}^k$), and $g(x,y,z) = 1$ otherwise. Calculate the individual influences and the total influence of g.
   (c) Let $m = k$ and $\epsilon = 2^{-k}/100$. Show that if $h\colon \{-1,1\}^{2^m+m+k} \to \{-1,1\}$ is ϵ-close to g (that is, $\Pr[g \neq h] \leq \epsilon$) then h depends on $2^{\Omega(\operatorname{Inf}[g]/\epsilon)}$ variables.