Boolean Function Analysis — Assignment 1

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April 4, 2021

1. Let $f, g\colon \{\pm 1\}^n \to \mathbb{R}$ and $c \in \mathbb{R}$. Express the Fourier expansions of $cf, f+g, fg$ in terms of the Fourier expansions of f and g. That is, for $h \in \{cf, f+g, fg\}$ and $S \subseteq [n]$, give a formula for $\hat{h}(S)$ in terms of the Fourier coefficients of f and g.

2. The majority function on $n = 2m+1$ inputs is a function from $\{\pm 1\}^n$ to $\{\pm 1\}$ given by the formula
$$\mathrm{Maj}_n(x_1, \ldots, x_n) = \mathrm{sgn}(x_1 + \cdots + x_n).$$
Determine the Fourier expansions of $\mathrm{Maj}_1, \mathrm{Maj}_3, \mathrm{Maj}_5$ (we have seen one of these in class).

3. In this question we outline an alternative argument for classifying all polymorphisms of Maj_3. Suppose that $f\colon \{\pm 1\}^n \to \{\pm 1\}$ satisfies
$$f(\mathrm{Maj}_3(x_1, y_1, z_1), \ldots, \mathrm{Maj}_3(x_n, y_n, z_n)) = \mathrm{Maj}_3(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), f(z_1, \ldots, z_n))$$
for all $x, y, z \in \{\pm 1\}^n$.
   (a) What is the Fourier expansion of $f(\mathrm{Maj}_3(x_1, y_1, z_1), \ldots, \mathrm{Maj}_3(x_n, y_n, z_n))$?
   (b) What is the Fourier expansion of $\mathrm{Maj}_3(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), f(z_1, \ldots, z_n))$?
   (c) Compare the constant coefficients (the coefficients of the empty monomial 1) to deduce that $\mathbb{E}[f] \in \{0, \pm 1\}$, and so either $f = \pm 1$ or $\mathbb{E}[f] = 0$ (this is identical to what we did in class).
   (d) Assume that $\mathbb{E}[f] = 0$, and let $S \subseteq [n]$ be non-empty. Compare the coefficients of x_S in both Fourier expansions to conclude that either $|S| = 1$ or $\hat{f}(S) = 0$.
   (e) Conclude that either $f = \pm 1$ or $f = \pm x_i$ (this is identical to what we did in class).

4. In this question you will extend the analysis of linearity testing from the test $f(xy) = f(x)f(y)$ to the test $f(xyz) = f(x)f(y)f(z)$.
   (a) Express $\Pr_{x,y,z}[f(xyz) = f(x)f(y)f(z)]$ in terms of the Fourier coefficients of f.
   (b) Determine all functions $f\colon \{\pm 1\}^n \to \{\pm 1\}$ satisfying $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in \{\pm 1\}^n$.
   (c) Show that if $f\colon \{\pm 1\}^n \to \{\pm 1\}$ satisfies
$$\Pr_{x,y,z}[f(xyz) \neq f(x)f(y)f(z)] \leq \epsilon$$
then $\Pr[f \neq g] = O(\epsilon)$ for some g which satisfies $g(xyz) = g(x)g(y)g(z)$ for all $x, y, z \in \{\pm 1\}^n$.

5. In class we gave a list of all functions $f\colon \{\pm 1\}^n \to \{\pm 1\}$ which have degree at most 1. What are all the functions $f\colon \{\pm 1\}^n \to \{0, \pm 1\}$ which have degree at most 1?