Assignment 2*

Another way of proving hypercontractivity (under the uniform measure) is through the log Sobolev inequality for the hypercube $\{-1,1\}^n$:

$$\mathrm{Ent}[f^2] := \mathbb{E}[f^2 \log(f^2)] - \mathbb{E}[f^2] \log \mathbb{E}[f^2] \leq 2\,\mathrm{Inf}[f]. \tag{$*$}$$

This inequality holds for all $f\colon \{-1,1\}^n \to \mathbb{R}$. Here log is the natural logarithm. We will show that the log Sobolev inequality is equivalent to the following hypercontractive estimate:

$$\|T_{e^{-t}} f\|_{1+e^{2t}} \leq \|f\|_2. \tag{$**$}$$

We will need the following definitions: $(L_i f)(x) = [f(x) - f(x \oplus i)]/2$, where $x \oplus i$ results from x by flipping the ith coordinate, and $L = \sum_{i=1}^n L_i$. (In class we didn't divide by 2.)

1. Properties of the Laplacian:

(a) Show that $\langle L_i f, L_i g\rangle = \langle L_i f, g\rangle$.

Solution: We have

$$\begin{aligned}
\langle L_i f, L_i g\rangle &= \frac{1}{4}\,\mathbb{E}[(f(x) - f(x \oplus i))(g(x) - g(x \oplus i))] \\
&= \frac{1}{4}\,\mathbb{E}[(f(x) - f(x \oplus i))g(x)] + \frac{1}{4}\,\mathbb{E}[(f(x \oplus i) - f(x))g(x \oplus i)] = \langle L_i f, g\rangle.
\end{aligned}$$

(b) Show that $\langle Lf, f\rangle = \mathrm{Inf}[f]$.

Solution: We have

$$\langle Lf, f\rangle = \sum_i \langle L_i f, f\rangle = \sum_i \|L_i f\|^2 = \mathrm{Inf}[f].$$

(c) Show that $\frac{d}{dt} T_{e^{-t}} f = -L T_{e^{-t}} f$.

Solution: Notice that $L_i f = \sum_{i \in S} \hat{f}(S)\chi_S$ and so $Lf = \sum_S |S|\hat{f}(S)\chi_S$. Therefore

$$\frac{d}{dt} T_{e^{-t}} f = \frac{d}{dt} \sum_S e^{-t|S|}\hat{f}(S)\chi_S = -\sum_S e^{-t|S|}|S|\hat{f}(S)\chi_S = -LT_e^{-t} f.$$

2. Simple properties of the log Sobolev inequality:

(a) Prove that $\mathrm{Ent}[f^2] \geq 0$, and determine when $\mathrm{Ent}[f^2] = 0$.

Solution: Convexity of $\varphi(x) = x \log x$ shows that

$$\mathbb{E}[\varphi(f^2)] \geq \varphi(\mathbb{E}[f^2]),$$

which states that $\mathrm{Ent}[f^2] \geq 0$. Since $\varphi(x)$ is strictly convex, equality is possibly only if f^2 is constant.

(b) What happens to both sides of $(*)$ when f is multiplied by a constant?

Solution: Multiplying f by c has the following effect on the left-hand side:

$$\mathrm{Ent}[(cf)^2] = c^2\,\mathbb{E}[f^2(\log(f^2) + \log(c^2))] - c^2\,\mathbb{E}[f^2](\log \mathbb{E}[f^2] + \log(c^2)) = c^2\,\mathrm{Ent}[f^2].$$

Similarly, $\mathrm{Inf}[cf] = c^2\,\mathrm{Inf}[f]$.

(c) Show that $\mathrm{Ent}[(1+\epsilon f)^2] \sim 2\,\mathbb{V}[f]\epsilon^2$, and deduce the Poincaré inequality $\mathbb{V}[f] \leq \mathrm{Inf}[f]$ from $(*)$.

Solution: The Taylor expansion of $\psi(x) = x^2 \log x^2$ around $x = 1$ is

$$\psi(1+\epsilon) = 2\epsilon + 3\epsilon^2 + O(\epsilon^3).$$

Therefore

$$\mathbb{E}[\psi(1+\epsilon f)] = 2\epsilon\,\mathbb{E}[f] + 3\epsilon^2\,\mathbb{E}[f^2] + O(\epsilon)^3.$$

*This assignment is based on exercises 10.22, 10.23, 10.24, 10.26 from Ryan O'Donnell's *Analysis of Boolean functions*, and on Diaconis and Saloff-Coste, *Logarithmic Sobolev inequalities for finite Markov chains*.

Similarly, the Taylor expansion of $\varphi(x)$ around $x = 1$ is

$$\varphi(1+\epsilon) = \epsilon + \frac{1}{2}\epsilon^2 + O(\epsilon^3),$$

and so

$$\begin{aligned}\varphi(\mathbb{E}[(1+\epsilon f)^2]) &= \varphi(1 + 2\epsilon\,\mathbb{E}[f] + \epsilon^2\,\mathbb{E}[f^2]) \\ &= 2\epsilon\,\mathbb{E}[f] + \epsilon^2\,\mathbb{E}[f^2] + 2\epsilon^2\,\mathbb{E}[f]^2 + O(\epsilon^3).\end{aligned}$$

Put together, we have

$$\mathrm{Ent}[(1+\epsilon f)^2] = 2\epsilon^2\,\mathbb{V}[f] + O(\epsilon^3).$$

On the other hand,

$$\mathrm{Inf}[1+\epsilon f] = \epsilon^2 \sum_S |S|\hat{f}(S)^2 = \epsilon^2\,\mathrm{Inf}[f].$$

The Poincaré inequality follows.

(d) Show that if $(*)$ holds for all non-negative f then it holds for all f.

Solution: Let $g = |f|$. Then $\mathrm{Ent}[g^2] = \mathrm{Ent}[f^2]$ while

$$\mathrm{Inf}_i[g] = \frac{1}{4}\,\mathbb{E}[(|f(x)| - |f(x\oplus i)|)^2] \le \frac{1}{4}\,\mathbb{E}[(f(x) - f(x\oplus i))^2] = \mathrm{Inf}_i[f].$$

If $(*)$ holds for g then

$$\mathrm{Ent}[f] = \mathrm{Ent}[g] \le 2\,\mathrm{Inf}[g] \le 2\,\mathrm{Inf}[f].$$

3. Log Sobolev follows from hypercontractivity:
Let $F(t) = \|T_{e^{-t}} f\|_{p(t)}$, for an as yet unspecified $p(t)$ and a non-negative f.

(a) Let $G(t) = F(t)^{p(t)}$. Show that

$$G'(t) = -p(t)\langle LT_{e^{-t}} f, (T_{e^{-t}} f)^{p(t)-1}\rangle + \frac{p'(t)}{p(t)}\,\mathbb{E}[(T_{e^{-t}} f)^{p(t)} \log(T_{e^{-t}} f)^{p(t)}].$$

Solution: Substituting the formula

$$(a(t)^{b(t)})' = a(t)^{b(t)-1}a'(t)b(t) + a(t)^{b(t)}b'(t)\log a(t)$$

in the expression $G(t) = \mathbb{E}[(T_{e^{-t}} f)^{p(t)}]$ and using 1(c),

$$G'(t) = \mathbb{E}[-(T_{e^{-t}} f)^{p(t)-1}(LT_{e^{-t}} f)p(t) + (T_{e^{-t}} f)^{p(t)}p'(t)\log(T_{e^{-t}} f)].$$

We get the form in the exercise by multiplying the second term by $\frac{p(t)}{p(t)}$.

(b) Show that

$$F'(t) = F(t)^{1-p(t)}\left[-\langle LT_{e^{-t}} f, (T_{e^{-t}} f)^{p(t)-1}\rangle + \frac{p'(t)}{p(t)^2}\,\mathrm{Ent}[(T_{e^{-t}} f)^{p(t)}]\right].$$

Solution: Substituting the formula

$$(a(t)^{1/b(t)})' = a(t)^{1/b(t)-1}\left(\frac{a'(t)}{b(t)} - \frac{b'(t)}{b(t)^2}a(t)\log a(t)\right)$$

in the expression $F(t) = G(t)^{1/p(t)}$,

$$\begin{aligned}F'(t) = F(t)^{1-p(t)}\Bigg(&-\langle LT_{e^{-t}} f, (T_{e^{-t}} f)^{p(t)-1}\rangle + \frac{p'(t)}{p(t)^2}\,\mathbb{E}[(T_{e^{-t}} f)^{p(t)}\log(T_{e^{-t}} f)^{p(t)}] \\ &-\frac{p'(t)}{p(t)^2}\,\mathbb{E}[(T_{e^{-t}} f)^{p(t)}]\log\mathbb{E}[(T_{e^{-t}} f)^{p(t)}]\Bigg),\end{aligned}$$

which equals the desired expression.
(c) Let $p(t) = 1 + e^{2t}$. Show that $\frac{p'(t)}{p(t)^2} \le \frac{1}{2}$ for all $t \ge 0$.
Solution: The desired inequality reads $4e^{2t} \le (1 + e^{2t})^2$, which holds since $(1 - e^{2t})^2 \ge 0$.
(d) Show that $(**)$ implies that $F'(0) \le 0$, and deduce the log Sobolev inequality.
Solution: Equation $(**)$ states that $F(t) \le F(0)$, implying $F'(0) \le 0$. Since $p(0) = 2$ and $p'(0)/p(0)^2 = 1/2$, we get

$$\frac{1}{2}\,\mathrm{Ent}[f^2] \le \langle Lf, f\rangle = \mathrm{Inf}[f].$$

4. Hypercontractivity follows from log Sobolev:
(a) Show that for all $a, b \ge 0$ and $p \ge 2$,

$$(a^{p-1} - b^{p-1})(a - b) \ge \frac{4(p-1)}{p^2}(a^{p/2} - b^{p/2})^2.$$

Hint: Justify and use the inequality $(\frac{1}{a-b}\int_b^a t^{p/2-1}\,dt)^2 \le \frac{1}{a-b}\int_b^a t^{p-2}\,dt$ for $a > b \ge 0$.
Solution: The inequality in the hint follows from the Cauchy–Schwartz inequality. Calculating the integrals explicitly, we get

$$\frac{1}{(a-b)^2}\frac{4}{p^2}(a^{p/2} - b^{p/2})^2 \le \frac{1}{a-b}\frac{1}{p-1}(a^{p-1} - b^{p-1}).$$

Rearrangement gives the inequality.
(b) Show that for $p \ge 2$, $\langle L_i f, L_i(f^{p-1})\rangle \ge \frac{4(p-1)}{p^2}\langle L_i(f^{p/2}), L_i(f^{p/2})\rangle$, and deduce $\langle Lf, f^{p-1}\rangle \ge \frac{4(p-1)}{p^2}\langle Lf^{p/2}, f^{p/2}\rangle$.
Solution: We have

$$\begin{aligned}
\langle L_i f, L_i(f^{p-1})\rangle &= \frac{1}{4}\,\mathbb{E}[(f(x) - f(x \oplus i))(f(x)^{p-1} - f(x \oplus i)^{p-1})] \\
&\ge \frac{4(p-1)}{p^2}\frac{1}{4}\,\mathbb{E}[(f(x)^{p/2} - f(x \oplus i)^{p/2})^2] \\
&= \frac{4(p-1)}{p^2}\langle L_i(f^{p/2}), L_i(f^{p/2})\rangle.
\end{aligned}$$

This shows the first inequality. The second follows from 1(a).
(c) Show that the log Sobolev inequality implies that $F'(t) \le 0$ (see previous exercise), and deduce $(**)$.
Solution: The log Sobolev inequality shows that

$$\frac{p'(t)}{p(t)^2}\,\mathrm{Ent}[(T_{e^{-t}}f)^{p(t)}] = \frac{2(p(t)-1)}{p(t)^2}\,\mathrm{Ent}[(T_{e^{-t}}f)^{p(t)}] \le \frac{4p(t)-1}{p(t)^2}\,\mathrm{Inf}[(T_{e^{-t}}f)^{p(t)/2}].$$

Now 1(b) shows that

$$\frac{4p(t)-1}{p(t)^2}\,\mathrm{Inf}[(T_{e^{-t}}f)^{p(t)/2}] = \frac{4p(t)-1}{p(t)^2}\langle L(T_{e^{-t}}f)^{p(t)/2}, (T_{e^{-t}}f)^{p(t)/2}\rangle \le \langle LT_{e^{-t}}f, (T_{e^{-t}}f)^{p(t)-1}\rangle.$$

In view of 3(b), this shows that $F'(t) \le 0$. In particular, $F(t) \le F(0)$ for all $t \ge 0$, which is just a restatement of $(**)$.

5. Independent proof of log Sobolev:
(a) Let $g(t) = 2t^2 - \mathrm{Ent}[(1 + tx_1)^2]$ (here $n = 1$). Show that $g(0) = g'(0) = 0$ and $g''(t) \ge 0$ for $|t| < 1$, and deduce that $g(t) \ge 0$ for all $|t| \le 1$.
Solution: We can write explicitly

$$\begin{aligned}
g(t) &= 2t^2 - (1+t)^2\log(1+t) - (1-t)^2\log(1-t) + (1+t^2)\log(1+t^2) \\
g'(t) &= 4t + 2t\log\frac{1+t^2}{1-t^2} + 2\log\frac{1-t}{1+t} \\
g''(t) &= \frac{4t^2}{t^2+1} + 2\log\frac{1+t^2}{1-t^2}
\end{aligned}$$

The last expression is clearly non-negative for $|t| < 1$. This shows that $g'(t) \le 0$ for $t \le 0$ and so $g(t) \ge g(0)$ for $t \le 0$, and similarly $g'(t) \ge 0$ for $t \ge 0$ implying $g(t) \ge g(0)$ for $t \ge 0$. This shows that $g(0) \le g(t)$ for all $|t| \le 1$.

(b) Deduce the log Sobolev inequality for $n = 1$. Hint: replace f by $|f|/\mathbb{E}[|f|]$.

Solution: According to 2(b),2(d), we can assume that f is non-negative and $\mathbb{E}[f] = 1$. Since $\mathbb{E}[f] = 1$, f has the form $f(x) = 1 + tx$. Since $f \ge 0$, $|t| < 1$. The log Sobolev inequality follows from $g(t) \ge 0$.

(c) Show that for any two functions f, g on $\{-1, 1\}^n$ we have

$$\left(\sqrt{\mathbb{E}[f^2]} - \sqrt{\mathbb{E}[g^2]}\right)^2 \le \mathbb{E}[(f-g)^2].$$

Solution: This is a simple application of the Cauchy–Schwartz inequality:

$$(\sqrt{\mathbb{E}[f^2]} - \sqrt{\mathbb{E}[g^2]})^2 = \mathbb{E}[f^2] + \mathbb{E}[g^2] - 2\sqrt{\mathbb{E}[f^2]\,\mathbb{E}[g^2]} \le \mathbb{E}[f^2] + \mathbb{E}[g^2] - 2\,\mathbb{E}[fg] = \mathbb{E}[(f-g)^2].$$

(d) Deduce the general log Sobolev inequality. Hint: use induction on n, employing the one-dimensional log Sobolev inequality in the inductive step.

Solution: Let $f_+(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 1)$ and $f_-(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, -1)$. Assume inductively that $(*)$ holds for f_+, f_-. Then

$$\begin{aligned}
\mathrm{Ent}[f^2] &= \frac{1}{2}\,\mathbb{E}[f_+^2 \log(f_+^2)] + \frac{1}{2}\,\mathbb{E}[f_-^2 \log(f_-^2)] - \frac{\mathbb{E}[f_+^2] + \mathbb{E}[f_-^2]}{2} \log \frac{\mathbb{E}[f_+^2] + \mathbb{E}[f_-^2]}{2} \\
&= \frac{1}{2}\,\mathrm{Ent}[f_+^2] + \frac{1}{2}\,\mathrm{Ent}[f_-^2] + \frac{1}{2}\,\mathbb{E}[f_+^2] \log \mathbb{E}[f_+^2] + \frac{1}{2}\,\mathbb{E}[f_-^2] \log \mathbb{E}[f_-^2] - \frac{\mathbb{E}[f_+^2] + \mathbb{E}[f_-^2]}{2} \log \frac{\mathbb{E}[f_+^2] + \mathbb{E}[f_-^2]}{2}.
\end{aligned}$$

We can bound the first two terms by induction. For the remaining terms, define $h(1) = \sqrt{\mathbb{E}[f_+^2]}$ and $h(-1) = \sqrt{\mathbb{E}[f_-^2]}$, so that the remaining terms are $\mathrm{Ent}[h^2]$. Using one-dimensional log Sobolev inequality to bound these terms, we obtain

$$\begin{aligned}
\mathrm{Ent}[f^2] \le \mathrm{Inf}[f_+] + \mathrm{Inf}[f_-] + 2\,\mathrm{Inf}[h] &= \mathrm{Inf}[f_+] + \mathrm{Inf}[f_-] + \frac{\sqrt{\mathbb{E}[f_+^2]} - \sqrt{\mathbb{E}[f_-^2]}}{2} \\
&\le \mathrm{Inf}[f_+] + \mathrm{Inf}[f_-] + \frac{\mathbb{E}[(f_+ - f_-)^2]}{2}.
\end{aligned}$$

Finally,

$$2\,\mathrm{Inf}[f] = 2\sum_{i=1}^{n} \mathrm{Inf}_i[f] + 2\,\mathrm{Inf}_{n+1}[f] = \mathrm{Inf}[f_+] + \mathrm{Inf}[f_-] + \frac{\mathbb{E}[(f_+ - f_-)^2]}{2}.$$

This completes the proof.