Assignment 2*

Another way of proving hypercontractivity (under the uniform measure) is through the log Sobolev inequality for the hypercube $\{-1,1\}^n$:

$$\operatorname{Ent}[f^2] := \mathbb{E}[f^2 \log(f^2)] - \mathbb{E}[f^2] \log \mathbb{E}[f^2] \leq 2 \operatorname{Inf}[f]. \tag{$*$}$$

This inequality holds for all $f\colon \{-1,1\}^n \to \mathbb{R}$. Here log is the natural logarithm. We will show that the log Sobolev inequality is equivalent to the following hypercontractive estimate:

$$\|T_{e^{-t}} f\|_{1+e^{2t}} \leq \|f\|_2. \tag{$**$}$$

We will need the following definitions: $(L_i f)(x) = [f(x) - f(x \oplus i)]/2$, where $x \oplus i$ results from x by flipping the ith coordinate, and $L = \sum_{i=1}^n L_i$. (In class we didn't divide by 2.)

1. Properties of the Laplacian:
(a) Show that $\langle L_i f, L_i g \rangle = \langle L_i f, g \rangle$.
(b) Show that $\langle Lf, f \rangle = \operatorname{Inf}[f]$.
(c) Show that $\frac{d}{dt} T_{e^{-t}} f = -L T_{e^{-t}} f$.

2. Simple properties of the log Sobolev inequality:
(a) Prove that $\operatorname{Ent}[f^2] \geq 0$, and determine when $\operatorname{Ent}[f^2] = 0$.
(b) What happens to both sides of $(*)$ when f is multiplied by a constant?
(c) Show that $\operatorname{Ent}[(1+\epsilon f)^2] \sim 2\,\mathbb{V}[f]\epsilon^2$, and deduce the Poincaré inequality $\mathbb{V}[f] \leq \operatorname{Inf}[f]$ from $(*)$.
(d) Show that if $(*)$ holds for all non-negative f then it holds for all f.

3. Log Sobolev follows from hypercontractivity:
Let $F(t) = \|T_{e^{-t}} f\|_{p(t)}$, for an as yet unspecified $p(t)$ and a non-negative f.
(a) Let $G(t) = F(t)^{p(t)}$. Show that

$$G'(t) = -p(t) \langle L T_{e^{-t}} f, (T_{e^{-t}} f)^{p(t)-1} \rangle + \frac{p'(t)}{p(t)} \mathbb{E}[(T_{e^{-t}} f)^{p(t)} \log (T_{e^{-t}} f)^{p(t)}].$$

(b) Show that

$$F'(t) = F(t)^{1-p(t)} \left[-\langle L T_{e^{-t}} f, (T_{e^{-t}} f)^{p(t)-1} \rangle + \frac{p'(t)}{p(t)^2} \operatorname{Ent}[(T_{e^{-t}} f)^{p(t)}] \right].$$

(c) Let $p(t) = 1 + e^{2t}$. Show that $\frac{p'(t)}{p(t)^2} \leq \frac{1}{2}$ for all $t \geq 0$.
(d) Show that $(**)$ implies that $F'(0) \leq 0$, and deduce the log Sobolev inequality.

4. Hypercontractivity follows from log Sobolev:
(a) Show that for all $a, b \geq 0$ and $p \geq 2$,

$$(a^{p-1} - b^{p-1})(a-b) \geq \frac{4(p-1)}{p^2} (a^{p/2} - b^{p/2})^2.$$

Hint: Justify and use the inequality $(\frac{1}{a-b} \int_b^a t^{p/2-1}\, dt)^2 \leq \frac{1}{a-b} \int_b^a t^{p-2}\, dt$ for $a > b \geq 0$.
(b) Show that for $p \geq 2$, $\langle L_i f, L_i (f^{p-1}) \rangle \geq \frac{4(p-1)}{p^2} \langle L_i (f^{p/2}), L_i (f^{p/2}) \rangle$, and deduce $\langle Lf, f^{p-1} \rangle \geq \frac{4(p-1)}{p^2} \langle L f^{p/2}, f^{p/2} \rangle$.
(c) Show that the log Sobolev inequality implies that $F'(t) \leq 0$ (see previous exercise), and deduce $(**)$.

5. Independent proof of log Sobolev:
(a) Let $g(t) = 2t^2 - \operatorname{Ent}[(1+tx_1)^2]$ (here $n=1$). Show that $g(0) = g'(0) = 0$ and $g''(t) \geq 0$ for $|t| < 1$, and deduce that $g(t) \geq 0$ for all $|t| \leq 1$.
(b) Deduce the log Sobolev inequality for $n = 1$. Hint: replace f by $|f|/\mathbb{E}[|f|]$.
(c) Show that for any two functions f, g on $\{-1,1\}^n$ we have

$$\left(\sqrt{\mathbb{E}[f^2]} - \sqrt{\mathbb{E}[g^2]}\right)^2 \leq \mathbb{E}[(f-g)^2].$$

(d) Deduce the general log Sobolev inequality. Hint: use induction on n, employing the one-dimensional log Sobolev inequality in the inductive step.

*This assignment is based on exercises 10.22, 10.23, 10.24, 10.26 from Ryan O'Donnell's *Analysis of Boolean functions*, and on Diaconis and Saloff-Coste, *Logarithmic Sobolev inequalities for finite Markov chains*.