Assignment 2*

Another way of proving hypercontractivity (under the uniform measure) is through the log Sobolev inequality for the hypercube \( \{-1, 1\}^n \):

\[
\text{Ent}[f^2] := \mathbb{E}[f^2 \log(f^2)] - \mathbb{E}[f^2] \log \mathbb{E}[f^2] \leq 2 \text{Inf}[f].
\]  

(*)

This inequality holds for all \( f : \{-1, 1\}^n \to \mathbb{R} \). Here log is the natural logarithm. We will show that the log Sobolev inequality is equivalent to the following hypercontractive estimate:

\[
\|T_{2^{-t}} f\|_{1 + e^{2t}} \leq \|f\|_2.
\]  

(**)

We will need the following definitions: \( (L_i f)(x) = [f(x) - f(x \oplus i)]/2 \), where \( x \oplus i \) results from \( x \) by flipping the \( i \)-th coordinate, and \( L = \sum_{i=1}^n L_i \). (In class we didn’t divide by 2.)

1. Properties of the Laplacian:
   (a) Show that \( \langle L_i f, L_i g \rangle = \langle L_i f, g \rangle \).
   (b) Show that \( \langle L f, f \rangle = \text{Inf}[f] \).
   (c) Show that \( \frac{d}{dt} T_{2^{-t}} f = -L T_{2^{-t}} f \).

2. Simple properties of the log Sobolev inequality:
   (a) Prove that \( \text{Ent}[f^2] \geq 0 \), and determine when \( \text{Ent}[f^2] = 0 \).
   (b) What happens to both sides of (**) when \( f \) is multiplied by a constant?
   (c) Show that \( \text{Ent}[(1 + \epsilon f)^2] \sim 2 \mathbb{V}[f]^2 \), and deduce the Poincaré inequality \( \mathbb{V}[f] \leq \text{Inf}[f] \) from (*).
   (d) Show that if (*) holds for all non-negative \( f \) then it holds for all \( f \).

3. Log Sobolev follows from hypercontractivity:
   Let \( F(t) = \|T_{2^{-t}} f\|_{p(t)} \), for an as yet unspecified \( p(t) \) and a non-negative \( f \).
   (a) Let \( G(t) = F(t)^{p(t)} \). Show that
   \[
   G'(t) = -p(t) \langle L T_{2^{-t}} f, (T_{2^{-t}} f)^{p(t)-1} \rangle + \frac{p'(t)}{p(t)} \mathbb{E}[(T_{2^{-t}} f)^{p(t)} \log(T_{2^{-t}} f)^{p(t)}].
   \]
   (b) Show that
   \[
   F'(t) = F(t)^{1-p(t)} \left[-\langle L T_{2^{-t}} f, (T_{2^{-t}} f)^{p(t)-1} \rangle + \frac{p'(t)}{p(t)} \text{Ent}[(T_{2^{-t}} f)^{p(t)}] \right].
   \]
   (c) Let \( p(t) = 1 + e^{2t} \). Show that \( \frac{p'(t)}{p(t)} \leq \frac{1}{2} \) for all \( t \geq 0 \).
   (d) Show that (**) implies that \( F'(0) \leq 0 \), and deduce the log Sobolev inequality.

4. Hypercontractivity follows from log Sobolev:
   (a) Show that for all \( a, b \geq 0 \) and \( p \geq 2 \),
   \[
   (a^{p-1} - b^{p-1})(a - b) \geq \frac{4(p-1)}{p^2}(a^{p/2} - b^{p/2})^2.
   \]
   Hint: Justify and use the inequality \( \left( \frac{1}{\sqrt{\pi}} \int_a^b (p/2-1) t^{p/2-1} dt \right)^2 \leq \frac{1}{\sqrt{\pi}} \int_a^b t^{p-2} dt \) for \( a > b \geq 0 \).
   (b) Show that for \( p \geq 2 \), \( \langle L_i f, L_i (f^{p-1}) \rangle \geq \frac{4(p-1)}{p^2} \langle L_i (f^{p/2}), L_i (f^{p/2}) \rangle \), and deduce \( \langle L f, f^{p-1} \rangle \geq \frac{4(p-1)}{p^2} \langle L f^{p/2}, f^{p/2} \rangle \).
   (c) Show that the log Sobolev inequality implies that \( F'(t) \leq 0 \) (see previous exercise), and deduce (**).

5. Independent proof of log Sobolev:
   (a) Let \( g(t) = 2t^2 - \text{Ent}[(1 + t x_1)^2] \) (here \( n = 1 \)). Show that \( g(0) = g'(0) = 0 \) and \( g''(t) \geq 0 \) for \( |t| < 1 \), and deduce that \( g(t) \geq 0 \) for all \( |t| \leq 1 \).
   (b) Deduce the log Sobolev inequality for \( n = 1 \). Hint: replace \( f \) by \( |f|/\mathbb{E}[|f|] \).
   (c) Show that for any two functions \( f, g \) on \( \{-1, 1\}^n \) we have
   \[
   \left( \sqrt{\mathbb{E}[f^2]} - \sqrt{\mathbb{E}[g^2]} \right)^2 \leq \mathbb{E}[(f - g)^2].
   \]
   (d) Deduce the general log Sobolev inequality. Hint: use induction on \( n \), employing the one-dimensional log Sobolev inequality in the inductive step.

*This assignment is based on exercises 10.22, 10.23, 10.24, 10.26 from Ryan O’Donnell’s Analysis of Boolean functions, and on Diaconis and Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains.