Assignment 1

1. Let $f\colon \{0,1\}^n \to \mathbb{R}$, where we think of the input to f as a subset of $[n] = \{1,\ldots,n\}$. Consider the following operation W, which takes as input a function $\{0,1\}^n \to \mathbb{R}$, and produces another function $\{0,1\}^n \to \mathbb{R}$:

$$(Wf)(S) = \mathop{\mathbb{E}}_T[f(S \cup T) + f(S \cap T) - f(T)].$$

Here T is chosen uniformly among all subsets of $[n]$.

(a) A function f is *modular* if it is of the form $f(x_1,\ldots,x_n) = c_0 + \sum_{i=1}^n c_i x_i$. Show that every modular function is a fixed point of W.

(b) Calculate the Fourier expansion of Wf in terms of the Fourier expansion of f.

(c) Let $W^{(m)}f$ be the result of applying W to f, m times in a row. Show that as $m \to \infty$, $W^{(m)}f$ converges to some function $W^{(\infty)}f$, and that $W^{(\infty)}f$ is modular.

(d*) Find a formula for $W^{(\infty)}f$. (See Flavio Chierichetti, Abhimanyu Das, Anirban Dasgupta, Ravi Kumar, *Approximate Modularity*, Theorem 9.)

2. We can think of the Fourier expansion of a function $f\colon \{-1,1\}^n \to \mathbb{R}$ (with respect to the uniform measure) as an operator which takes a vector of length 2^n (the values of f) to another vector of length 2^n (the values of $\hat{f}$).

(a) Describe the matrix H of this operator. Do you recognize this matrix?

(b) Give a formula for computing f from its Fourier expansion.

(c) Suppose $f = g \cdot h$. Express $\hat{f}$ in terms of $\hat{g}, \hat{h}$.

(d) Show how to compute the Fourier expansion of f in time $O(n2^n)$.

3. The *sensitivity* of a $\{\pm 1\}$-valued function f at a point x is the number of coordinates i such that flipping x_i results in flipping the value of f.

(a) Show that the total influence of f equals its average sensitivity.

(b) Suppose that $g\colon \{-1,1\}^n \to \{-1,1\}$ can be written as a homogeneous multilinear polynomial of degree d (this means that all monomials of g have degree *exactly* d). Show that the total influence of g is exactly d.

(c) Show that the *maximal* sensitivity of g is d.

4. Consider the following continuous-time Markov chain on the set of states $\{-1,1\}^n$. The starting state is some given $x \in \{-1,1\}^n$. At each infinitesimal interval of length ϵ and for each $i \in [n]$ independently, there is a chance of ϵ of flipping the ith bit. (While this definition is informal, it can be formalized.)

(a) For fixed i, let $t_1, t_2, \ldots$ be the times at which bit i is flipped. Show that $t_1, t_2 - t_1, t_3 - t_2, \ldots$ are independent random variables distributed exponentially. (Hint: divide $[0,t]$ to $1/\epsilon$ intervals of length ϵ and take $\epsilon \to 0$ to calculate $\Pr[t_1 > t]$.)

(b) Show that the number of times bit i is flipped until time t has Poisson distribution.

(c) What is the distribution of the ith bit at time t?

(d) Let y_t be the state at time t, and define $(\mathbb{T}_t f)(x) = \mathbb{E}[f(y_t)]$. Relate $\mathbb{T}_t$ and the noise operator T_ρ.