Twenty (Simple) Questions

Joint work with
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Yuval Filmus, 26 April 2021, HUJI CS Colloquium

The Game of 20 Questions

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Bob

Thinks of a number between 1 to *n*

The Game of 20 Questions

Alice

Finds the number by asking Yes/No questions

Bob

Thinks of a number between 1 to n

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Thinks of a number between 1 to n

Cooperative!

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Finds the number by asking Yes/No questions

binary search: log*n* questions

Bob

Thinks of a number between 1 to *n*

Cooperative!

Distributional 20 Questions

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Bob

Samples a number between 1 to n **according to μ**

Distributional 20 Questions

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μ known to both parties!

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Huffman's algorithm: H(μ)+1 questions on average

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Samples a number between 1 to n **according to μ**

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Distributional 20 Questions

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Finds the number by asking Yes/No questions

H(μ) = entropy of μ = amortized # questions when solving many games in parallel

Huffman's algorithm: H(μ)+1 questions on average

Bob

Samples a number between 1 to n **according to μ**

μ known to both parties!

Distributional 20 Questions

Huffman's algorithm could involve complicated questions:

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Hunter Desportes

Is x one of 2,3,5,7,11,13?

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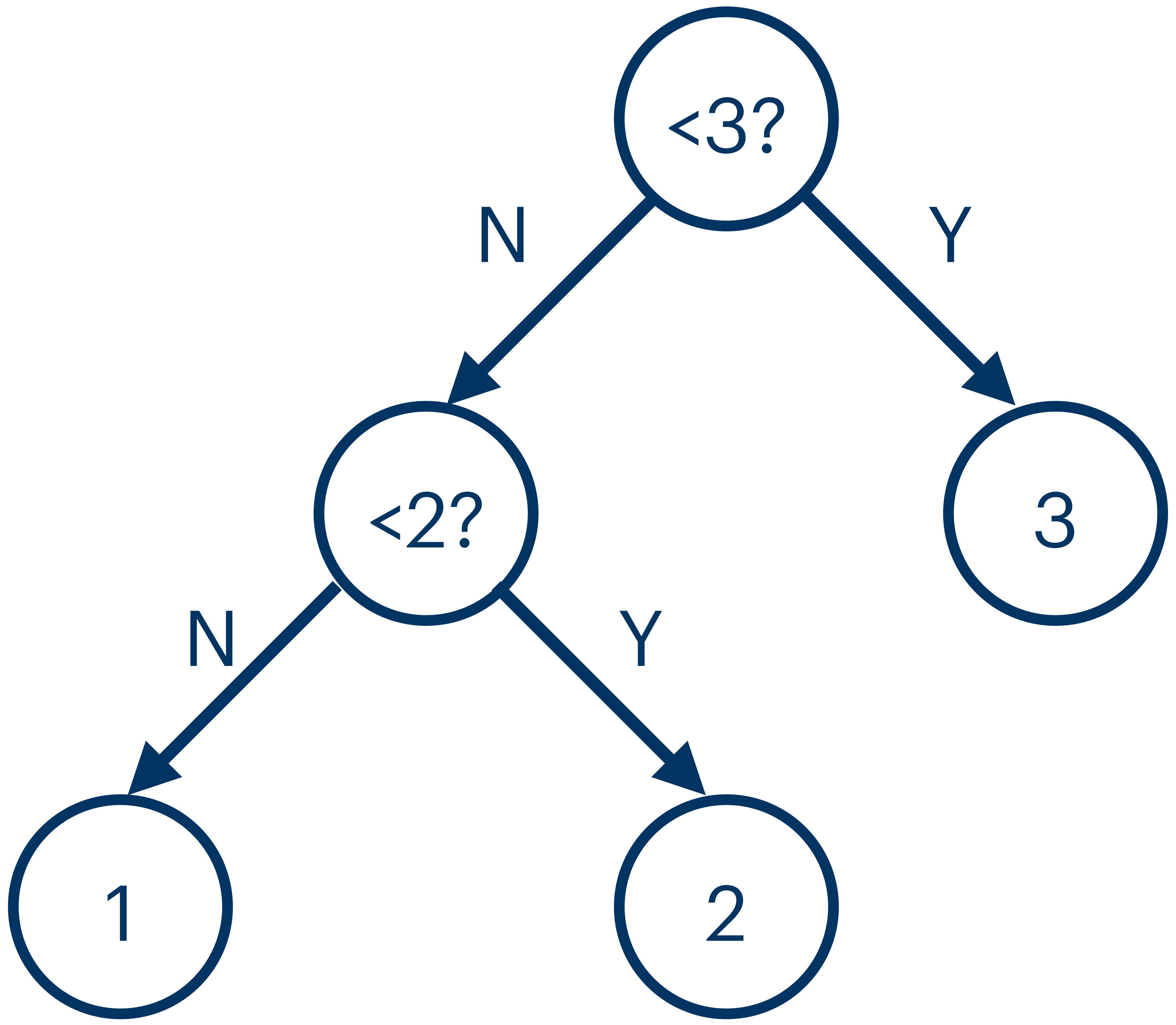
Is x one of
2,3,5,7,11,13?

What can we accomplish using simple questions?

Binary Search Trees



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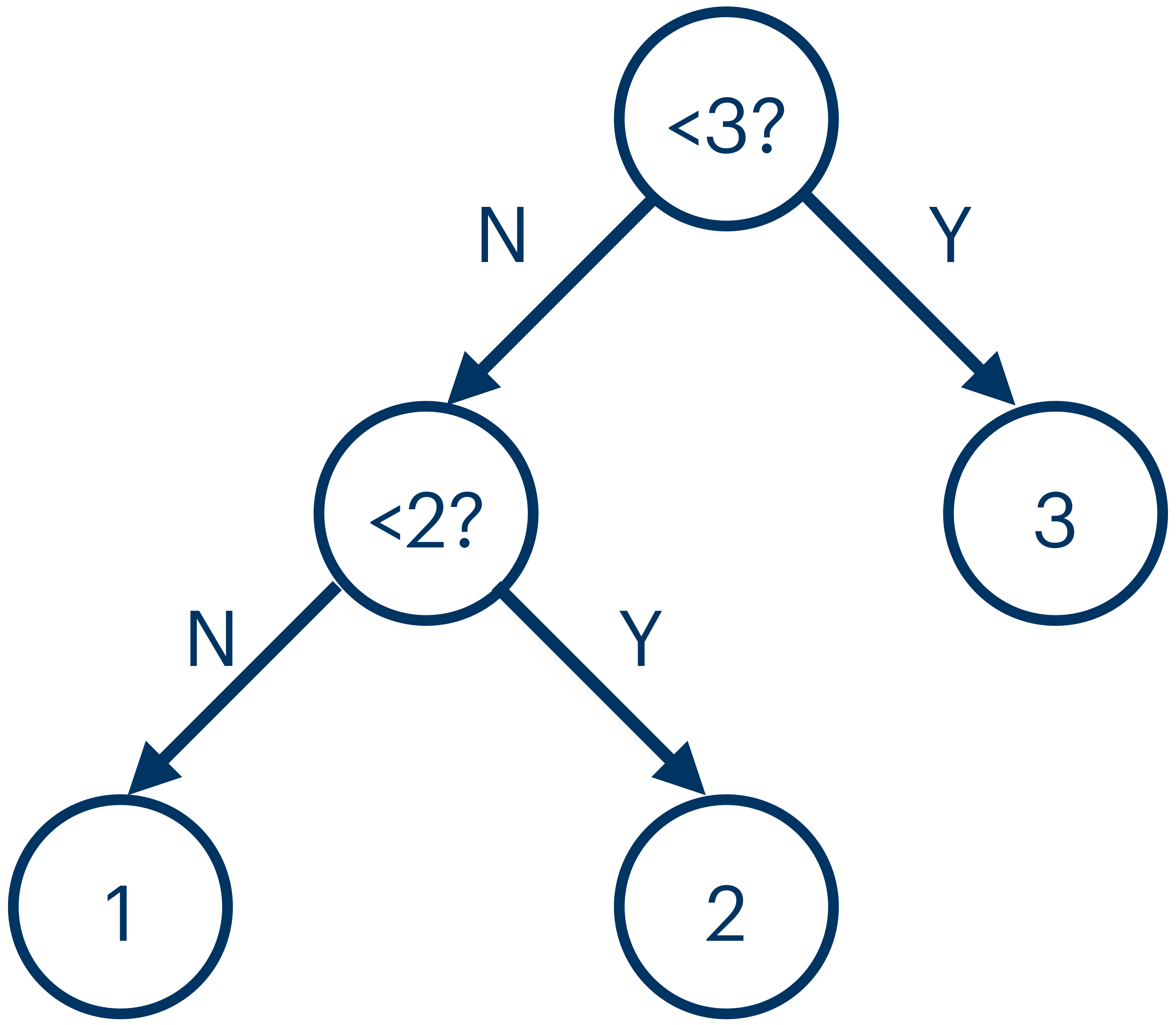

Gilbert and Moore: optimal BST achieves $H(\mu)+2$

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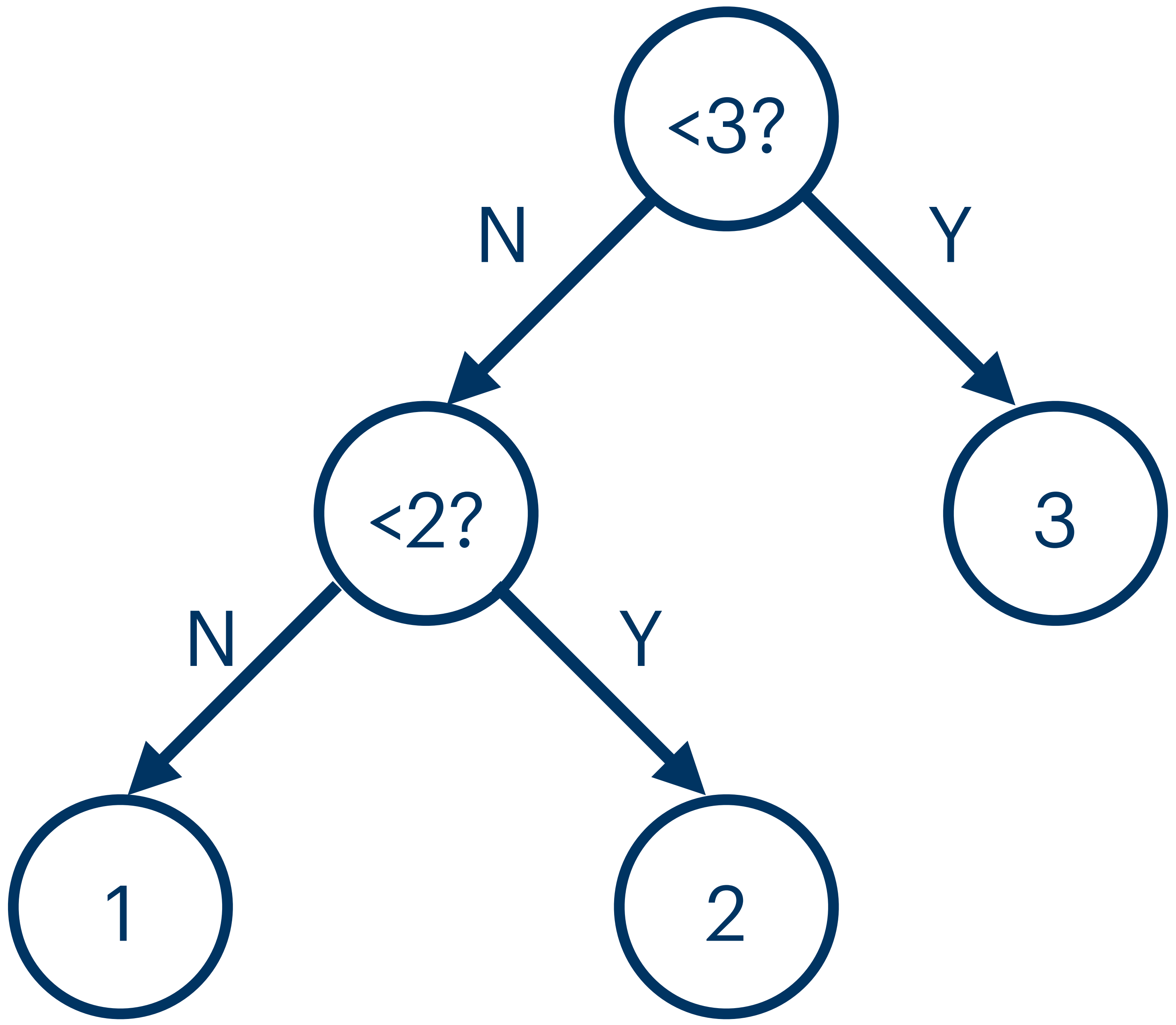
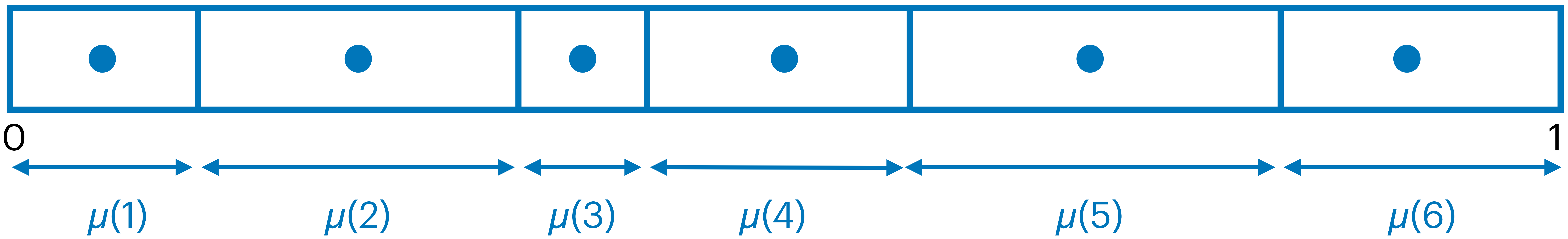


Gilbert and Moore: optimal BST achieves H(μ)+2

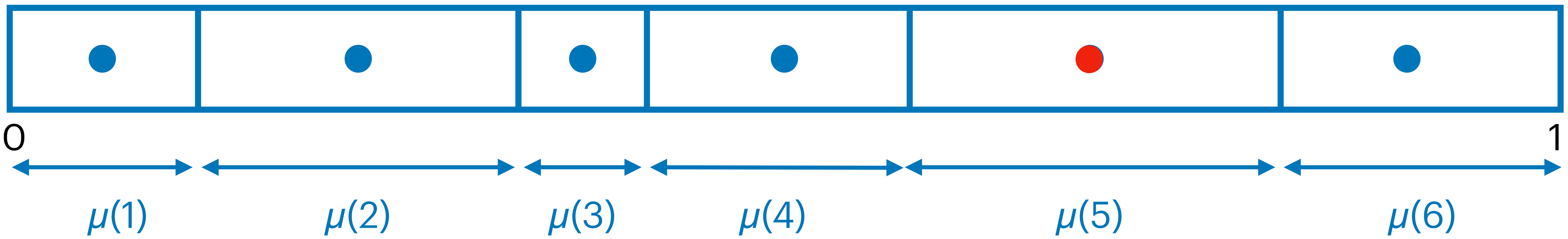
(optimal for distributions concentrated on some $x \in \{2,\ldots,n-1\}$)

Gilbert–Moore Algorithm



Binary search over [0,1]

Gilbert–Moore Algorithm



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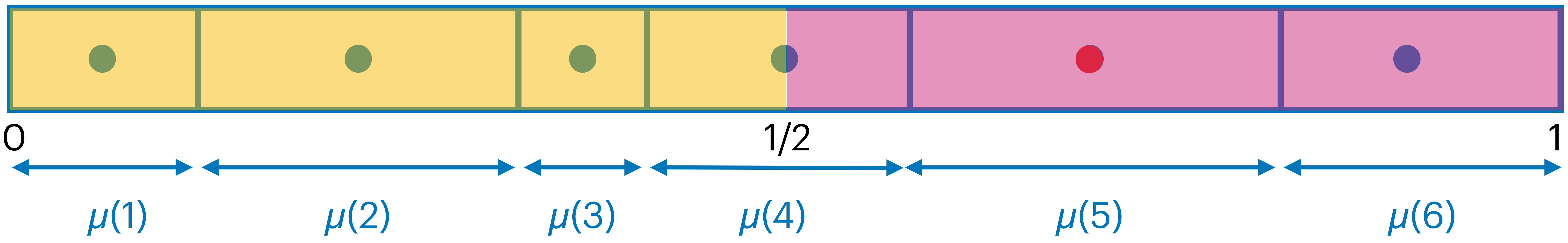

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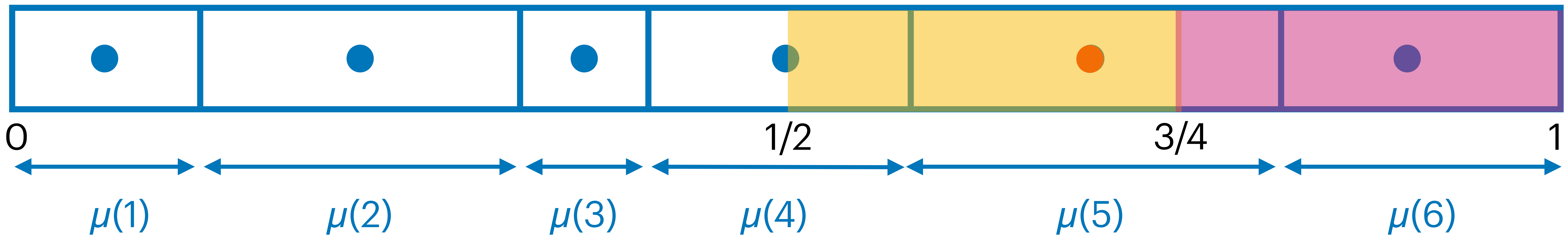

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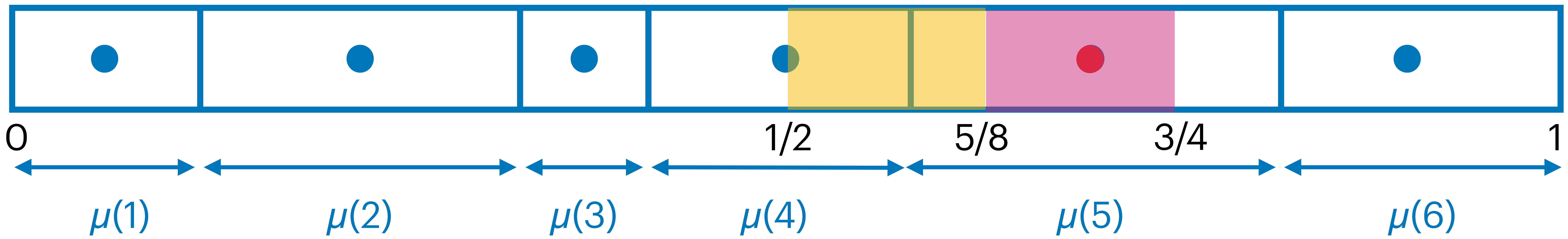

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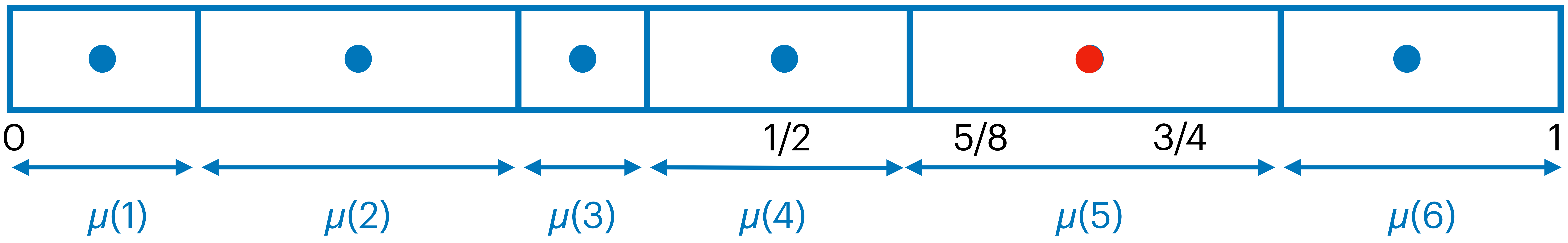

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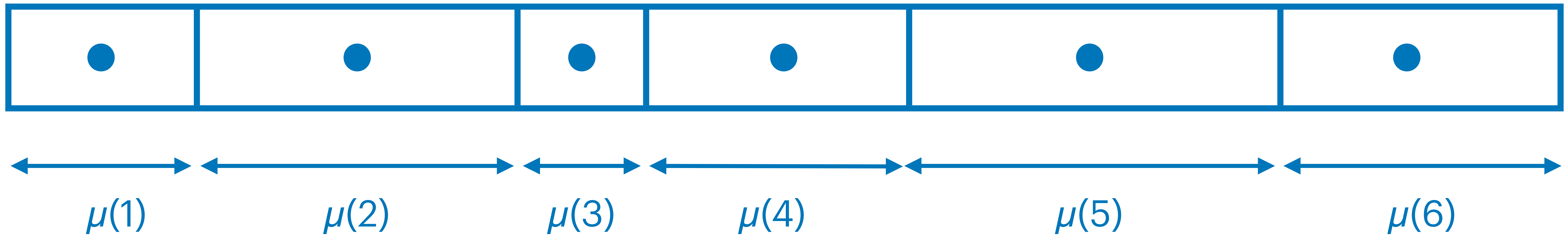
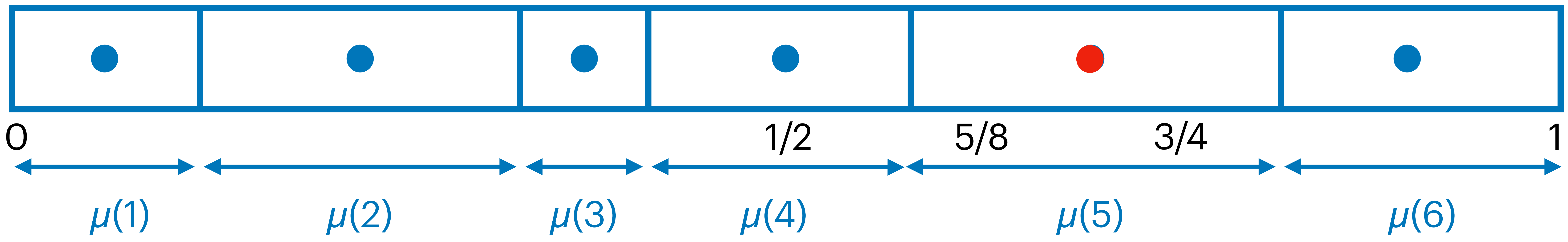
Rissanen–Horibe Algorithm



Ask most informative question

Gilbert–Moore Algorithm



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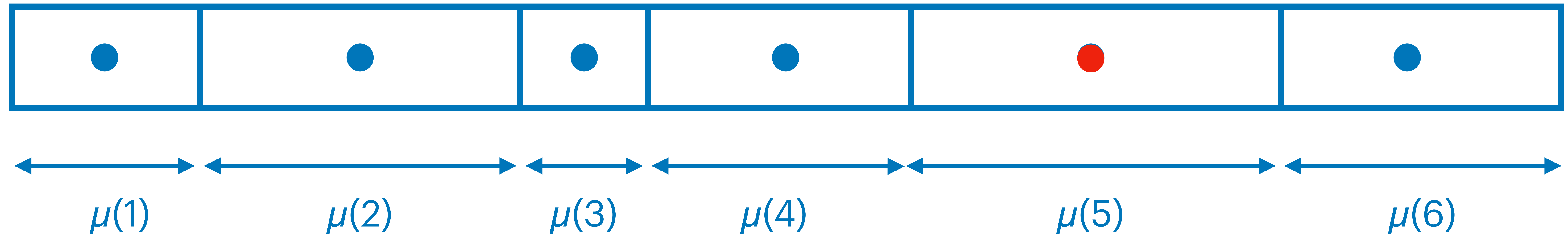

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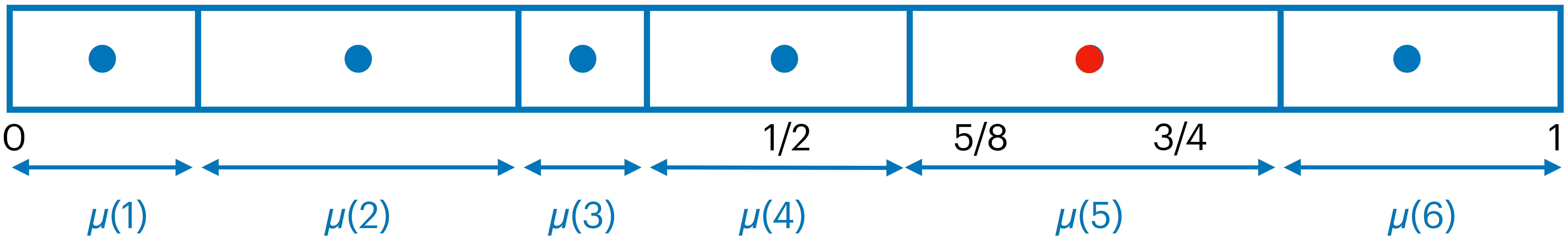

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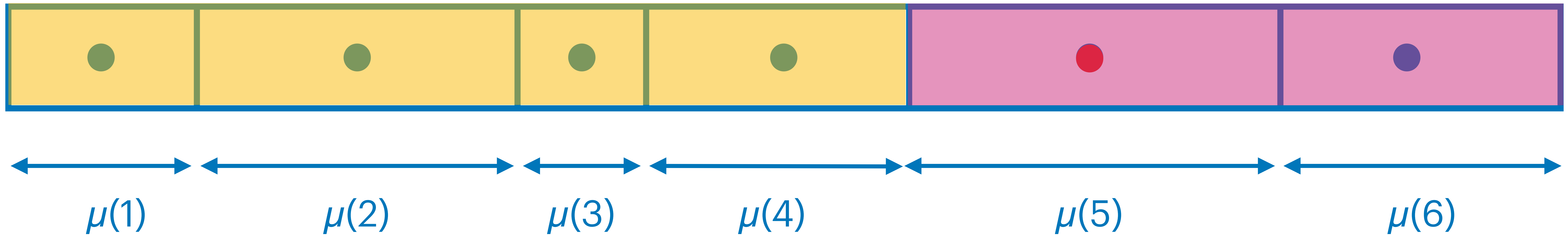

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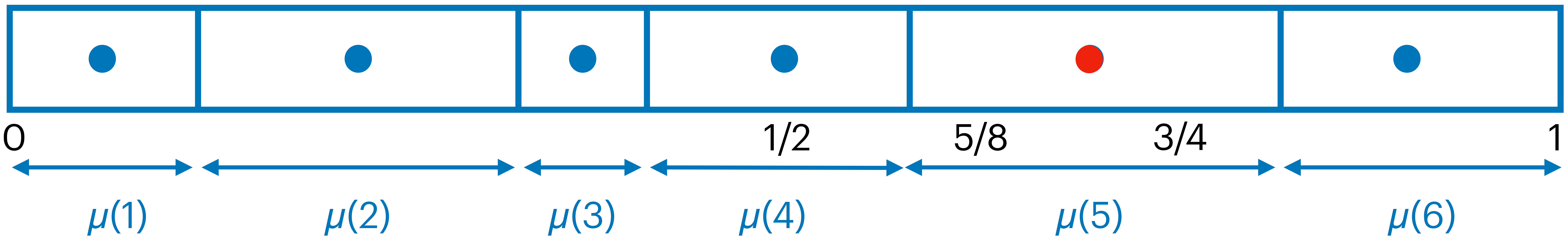

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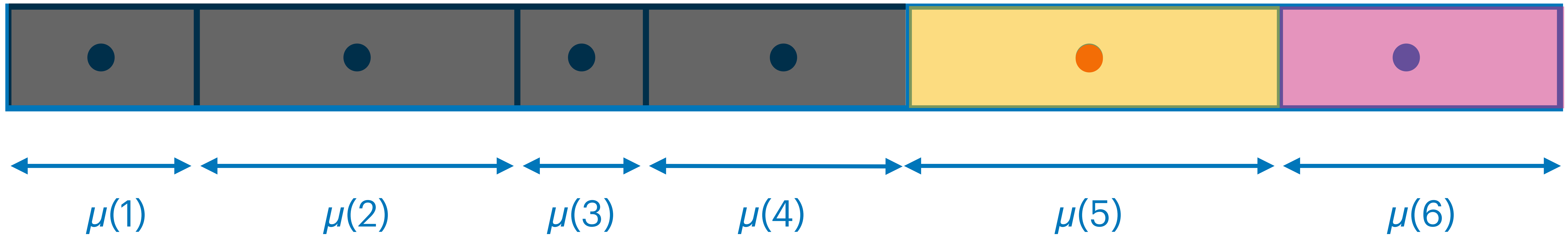
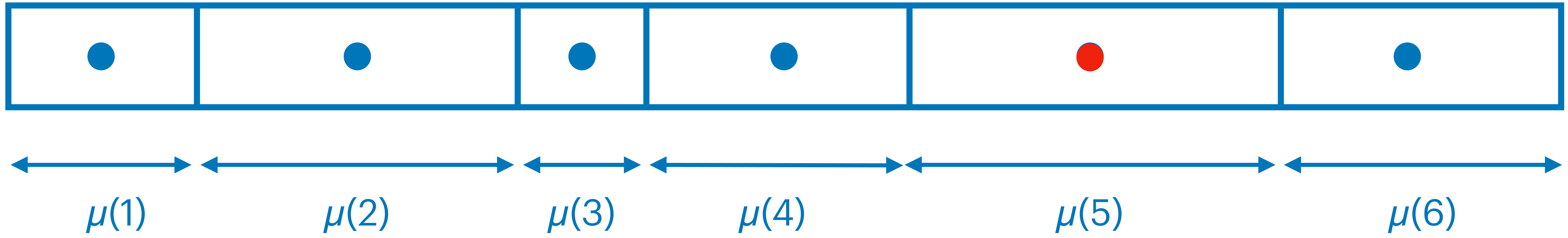


Ask most informative question

Gilbert–Moore Algorithm

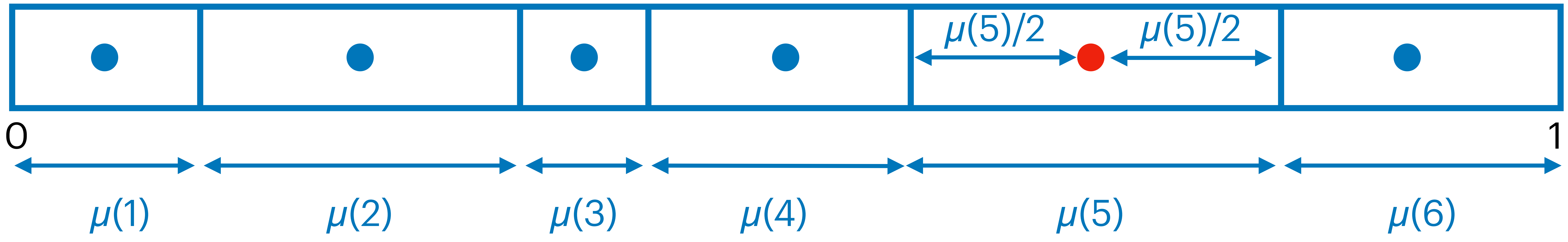
Both algorithms guarantee $H(\mu)+2$

Rissanen–Horibe Algorithm



Ask most informative question

Analysis of Gilbert–Moore



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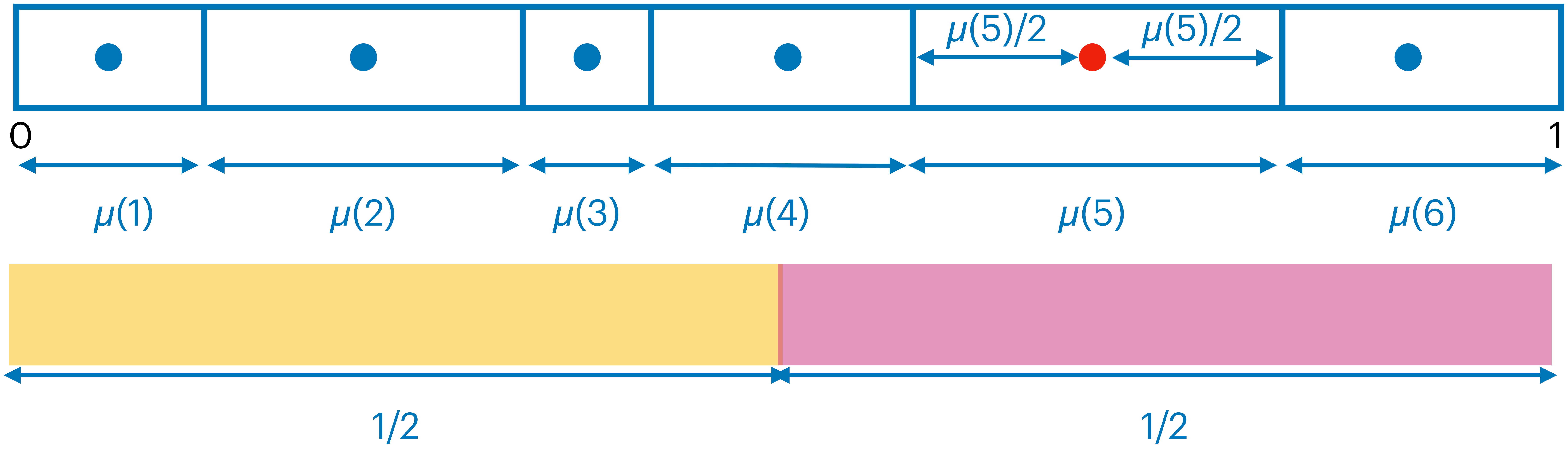

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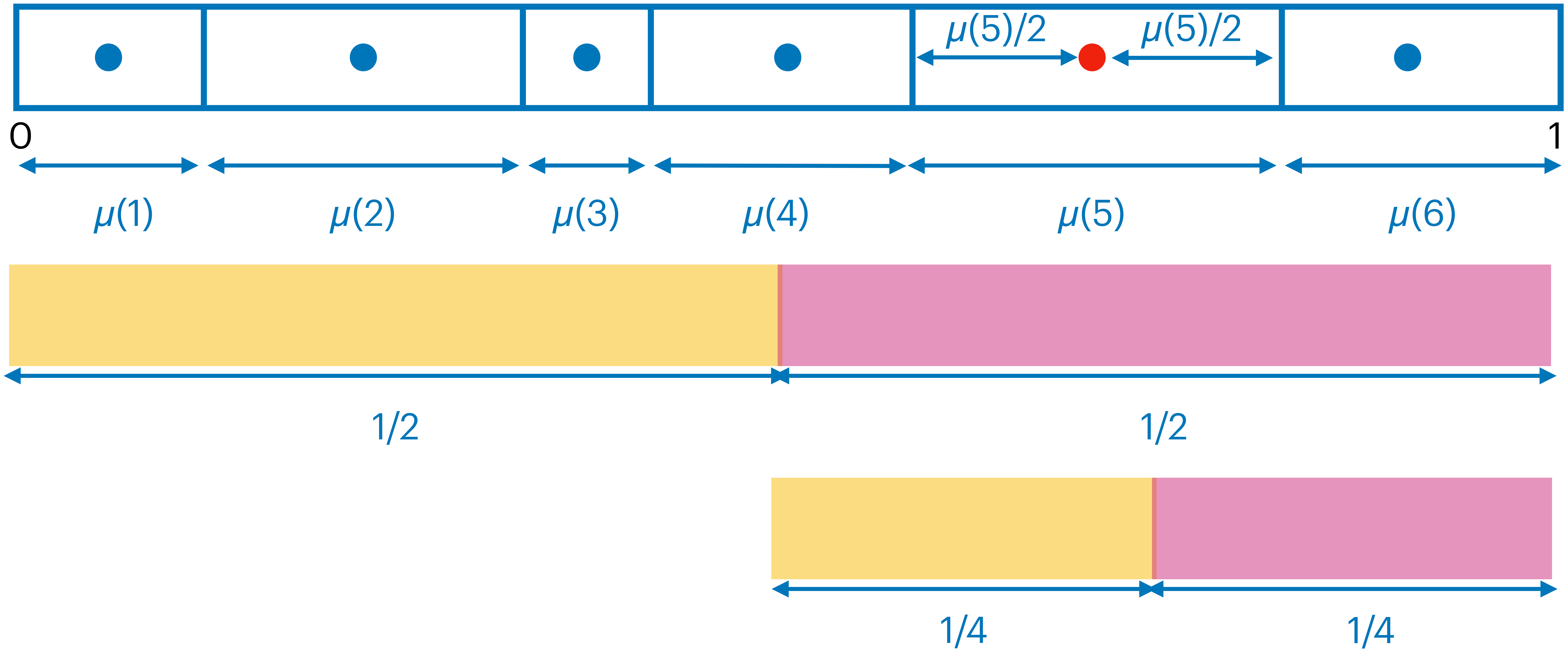

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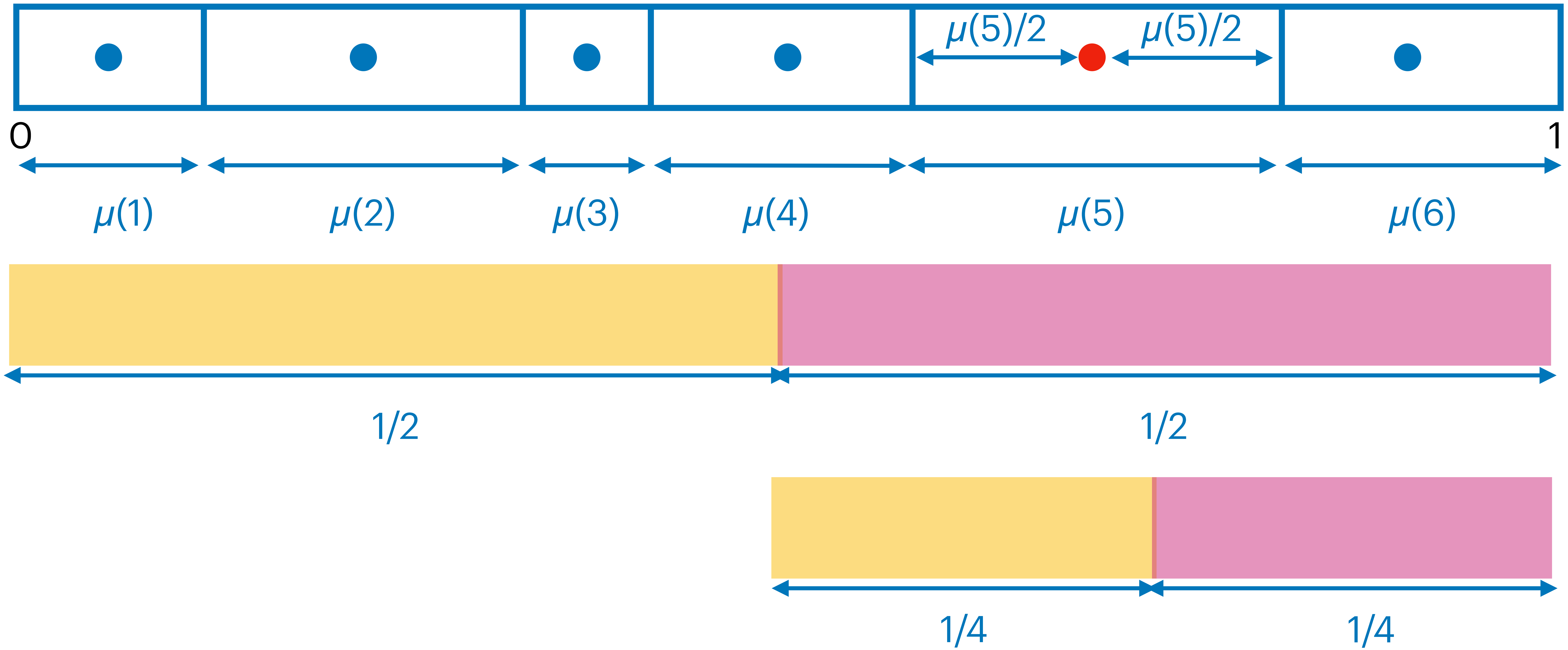
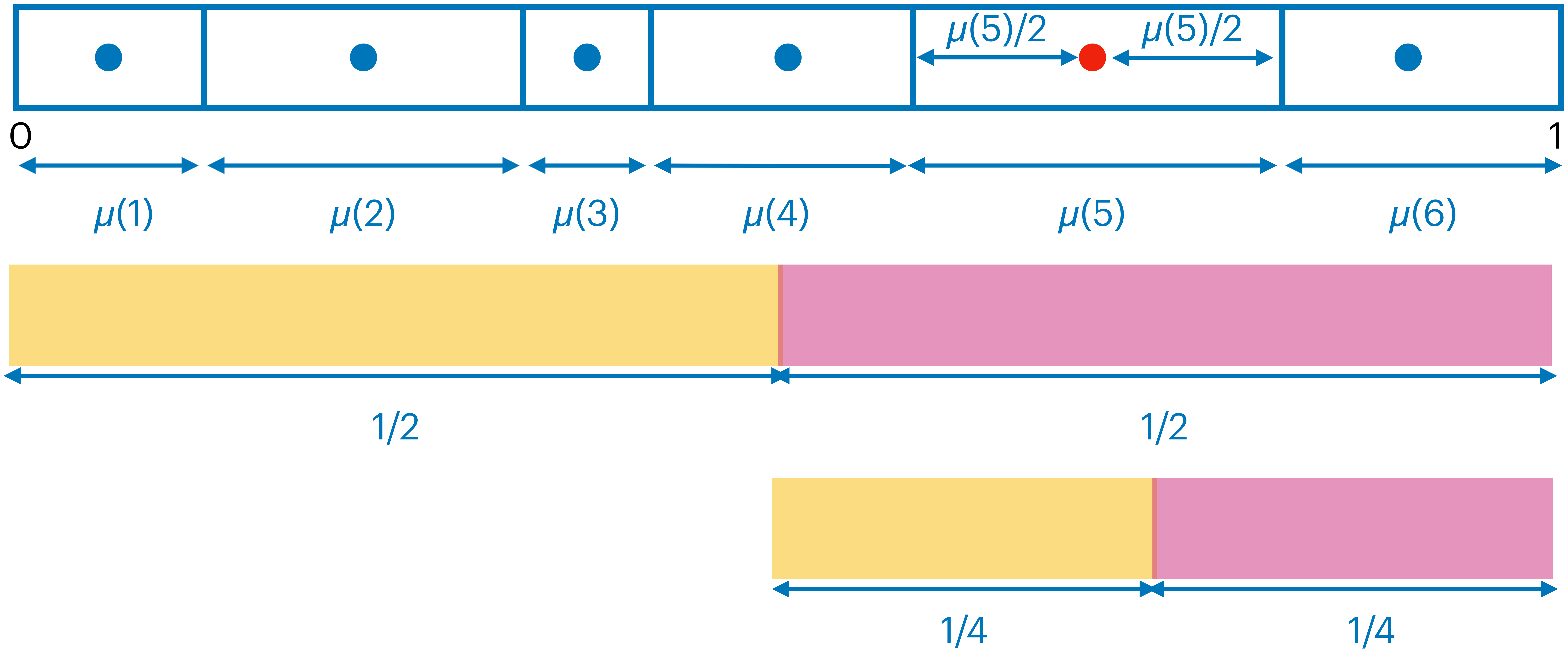


After k questions, zero in on interval of length 2^{-k}

Analysis of Gilbert–Moore



After k questions, zero in on interval of length 2^{-k}

Can stop once interval has length at most $\mu(x)/2$

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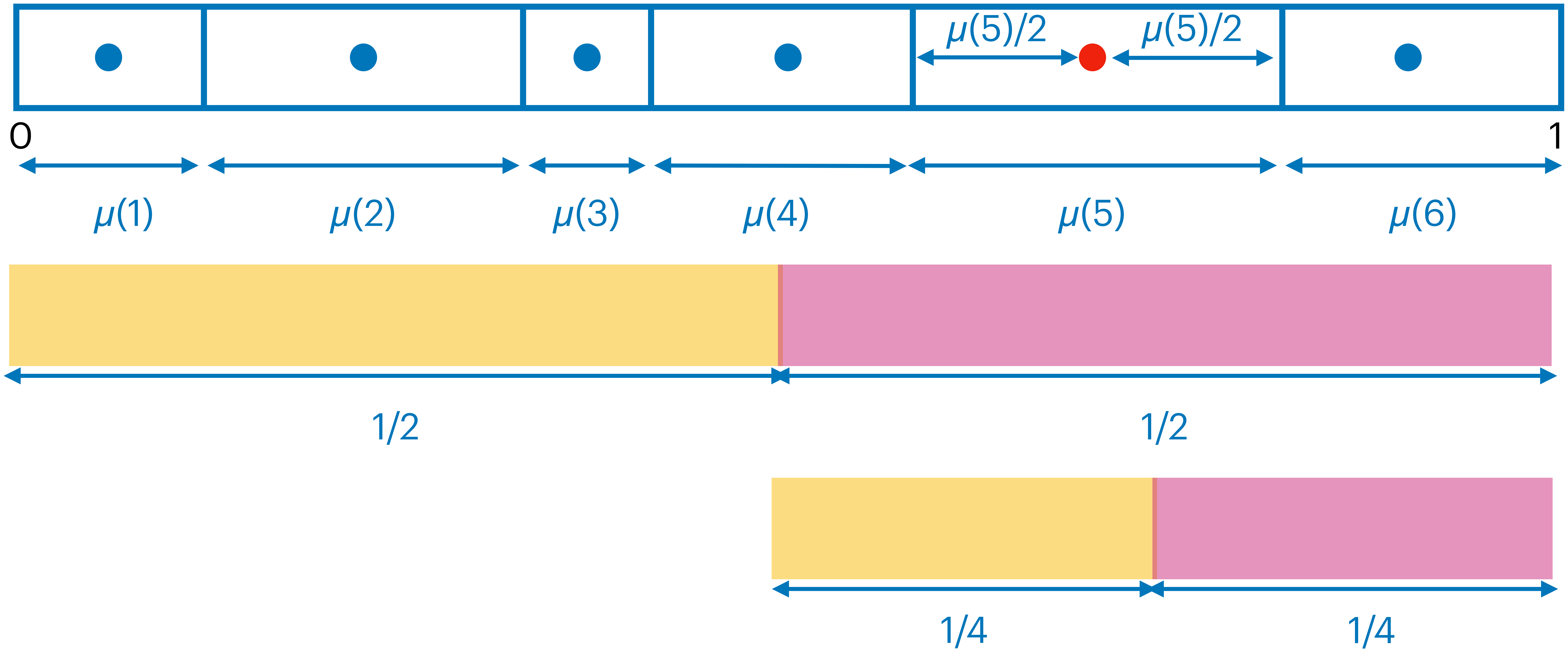

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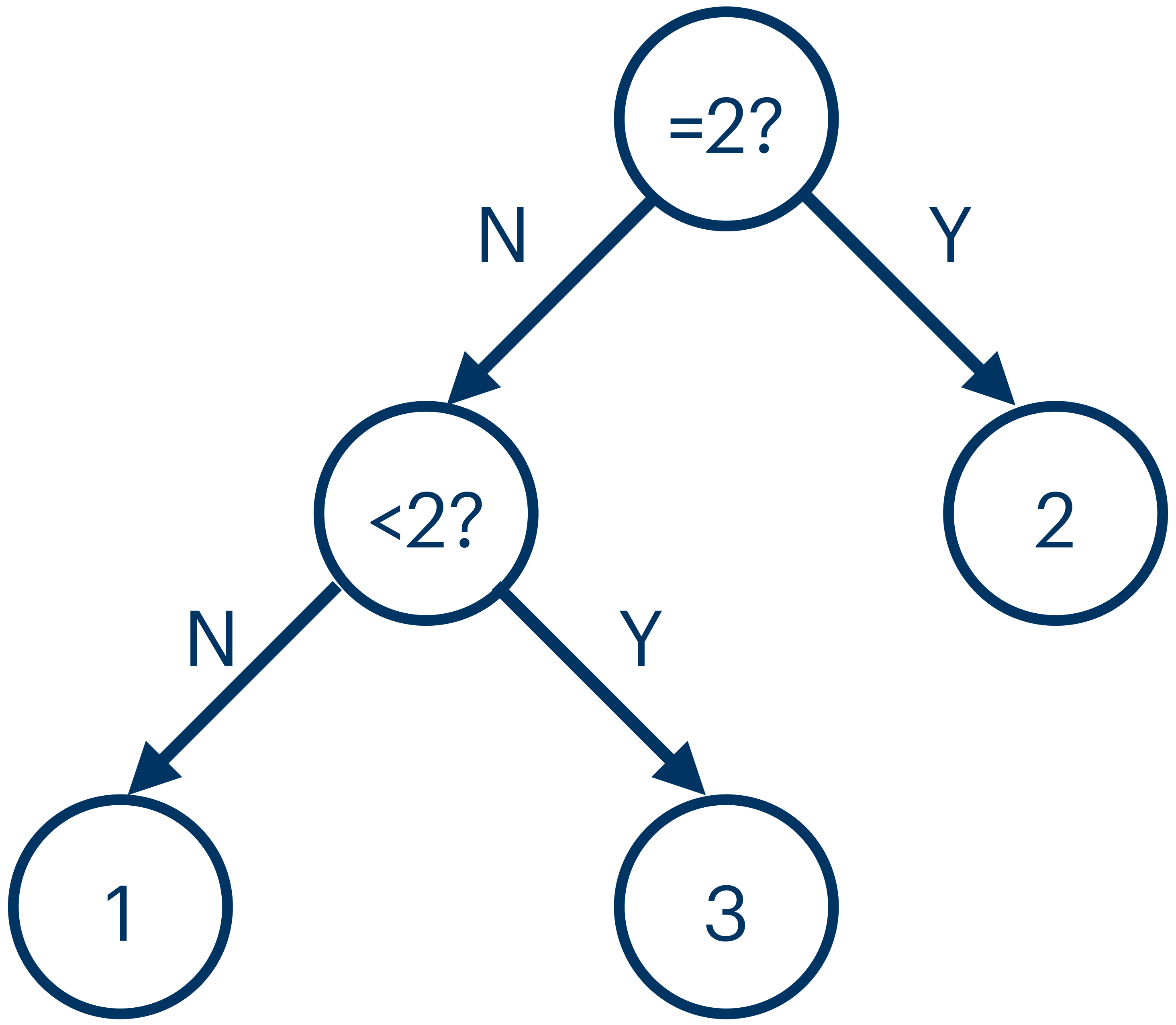
Stop after $\lceil \log(2/\mu(x)) \rceil < \log(1/\mu(x))+2$ questions

Binary Split Trees



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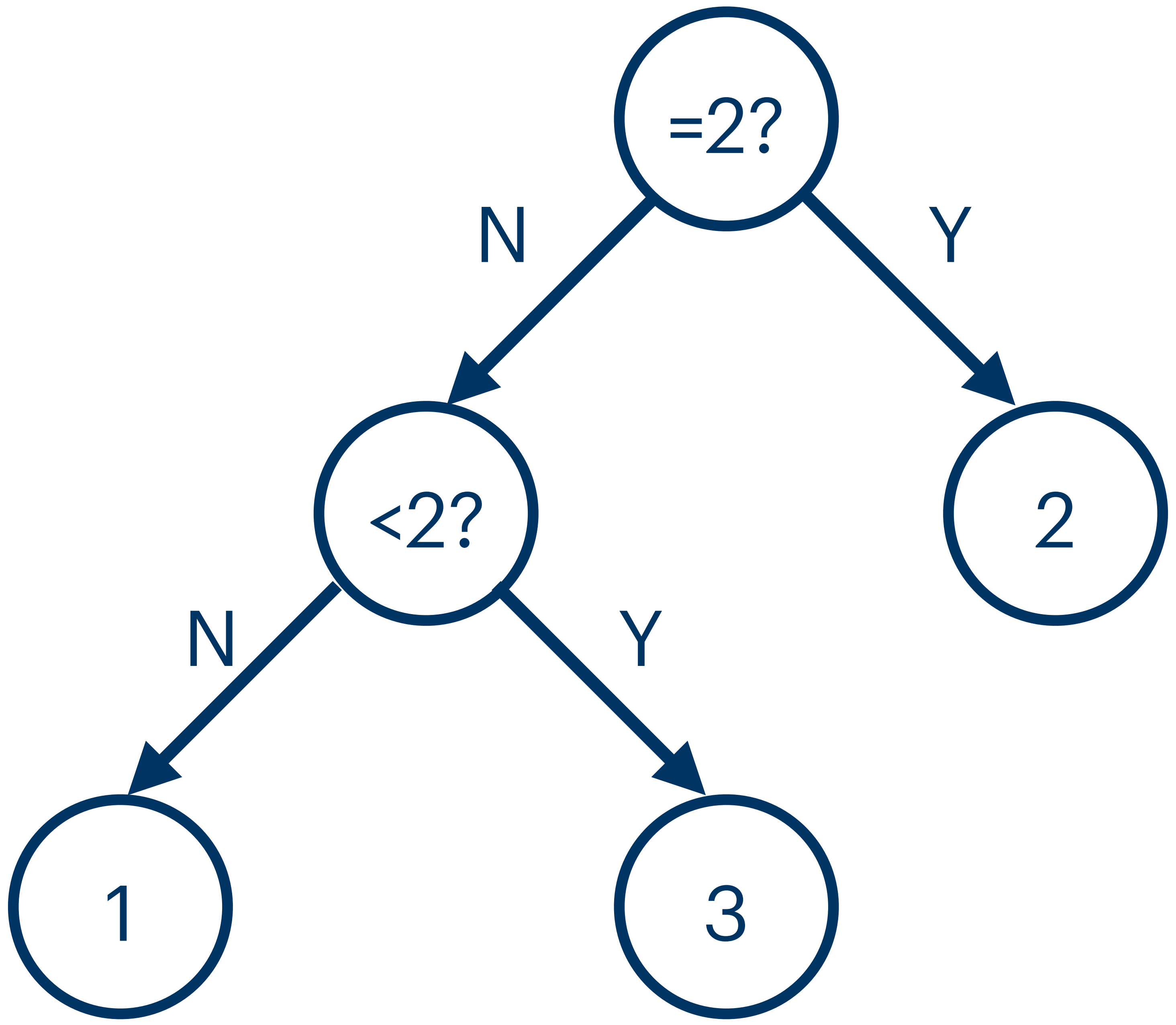
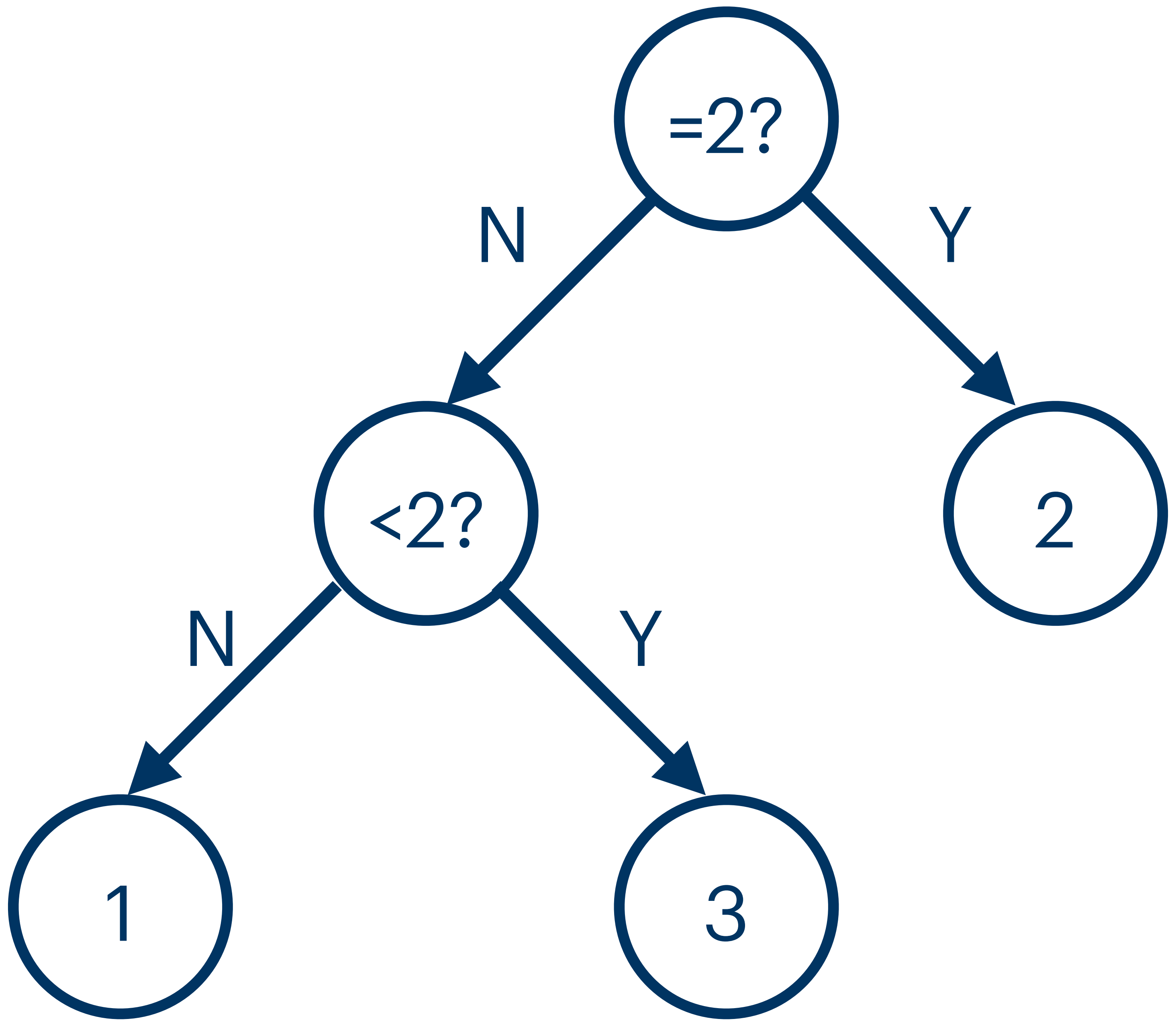


We show: optimal binary split tree achieves H(μ)+1

Binary Split Trees



We show: optimal binary split tree achieves H(μ)+1

Same performance guarantee as Huffman!

Our Algorithm

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If most probable element *i* has probability ≥0.3:

Ask if $x = i$

Otherwise:

Ask most informative “<” question

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1	2	3

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Why do we care?

Chunked Binary Split Trees

word 0	word 1	word 2	word 3

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=AB01?	<BDBB?	=0010?	<0042?
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Performance on *w* words: H(μ)+*w*

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Performance on w words: $H(\mu)+w$

Number of different questions: $2wn^{1/w}$

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Optimal for redundancy w!

Playing 20 Questions with a Liar

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Thinks of a number
between 1 to n

Playing 20 Questions with a Liar

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Finds the number by
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Bob allowed to lie k times

Playing 20 Questions with a Liar

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optimal cost:
log*n*+*k*loglog*n*

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Optimal cost: $H(\mu)+kH_2(\mu)$ on average

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$H(\mu) = E[\log 1/\mu]$
$H_2(\mu) = E[\log\log 1/\mu]$

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Optimal cost:
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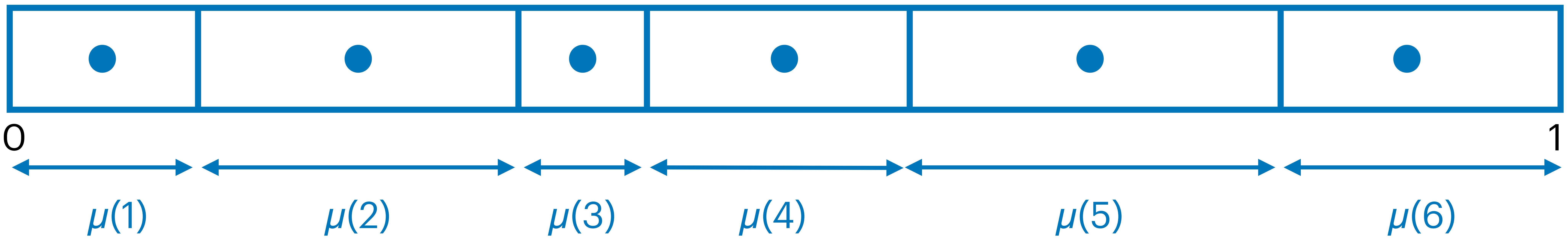
Gilbert–Moore with Lies



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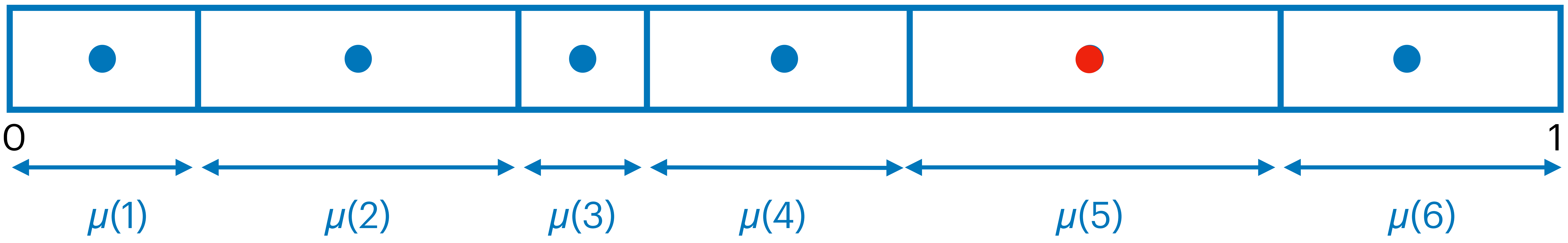

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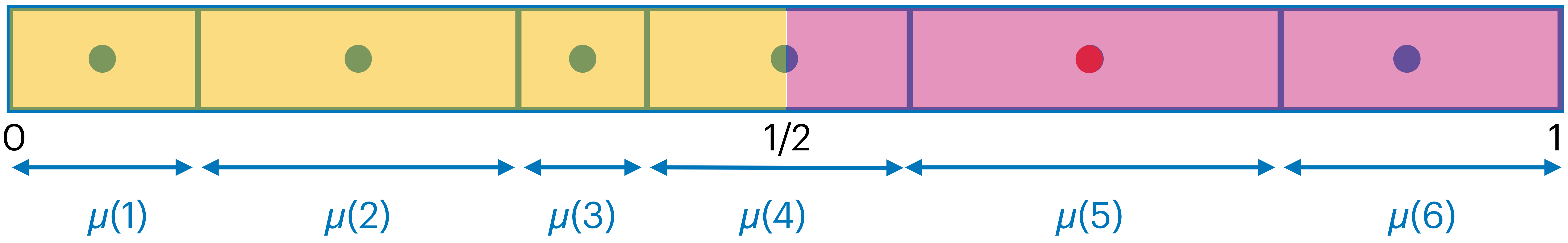

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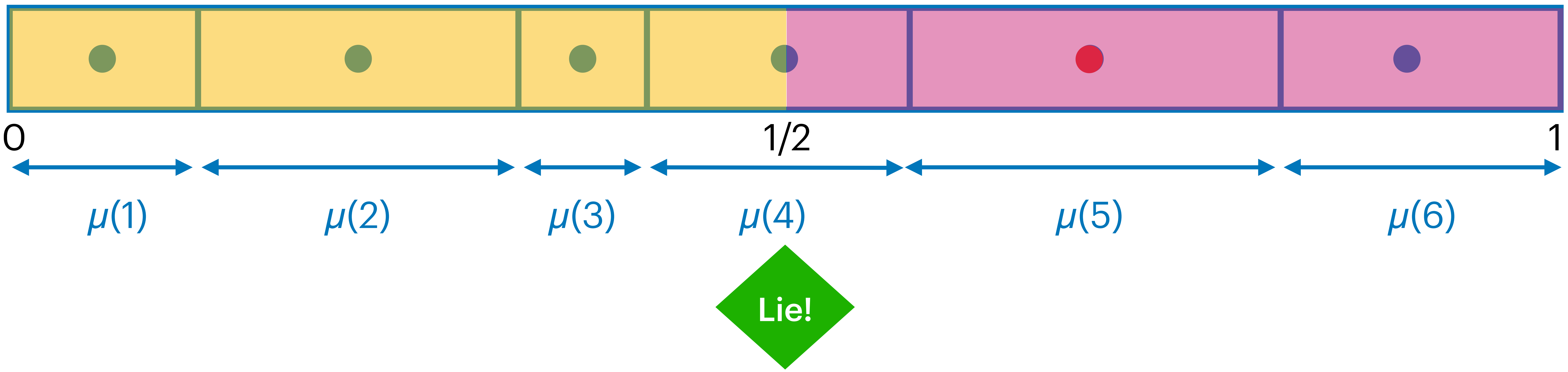

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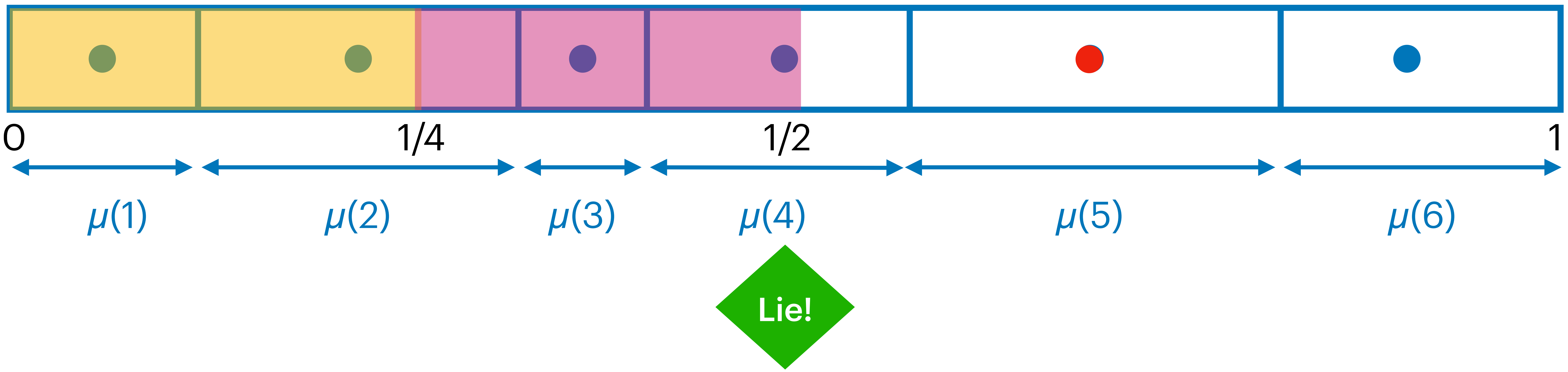

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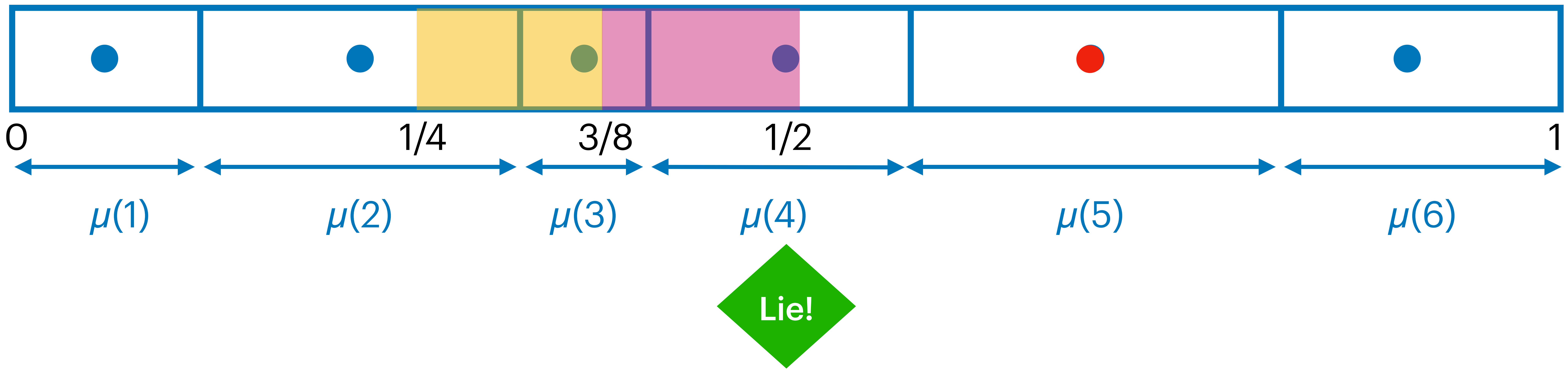

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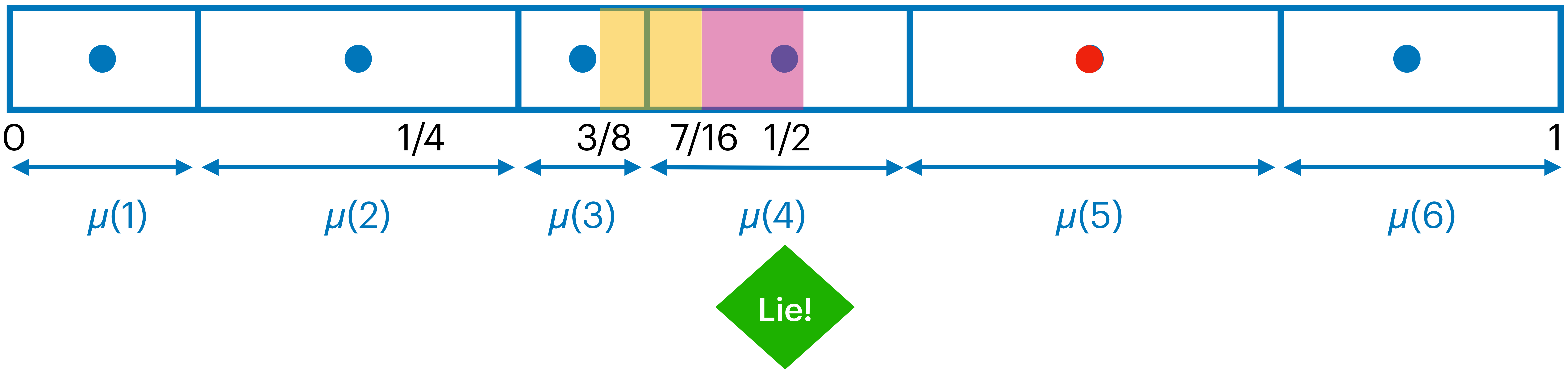

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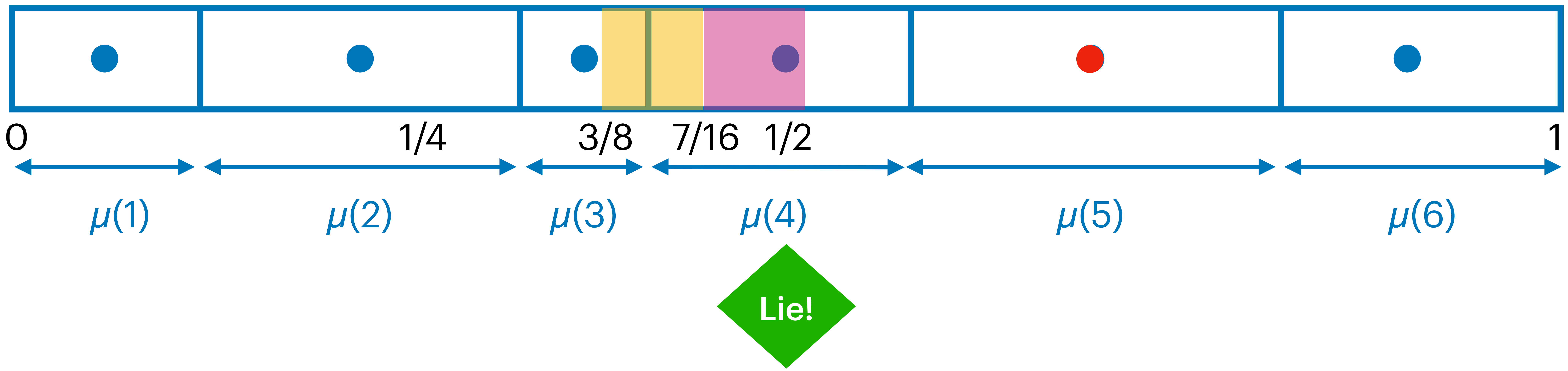

After first lie, answer always “>” – suspicious!

Gilbert–Moore with Lies

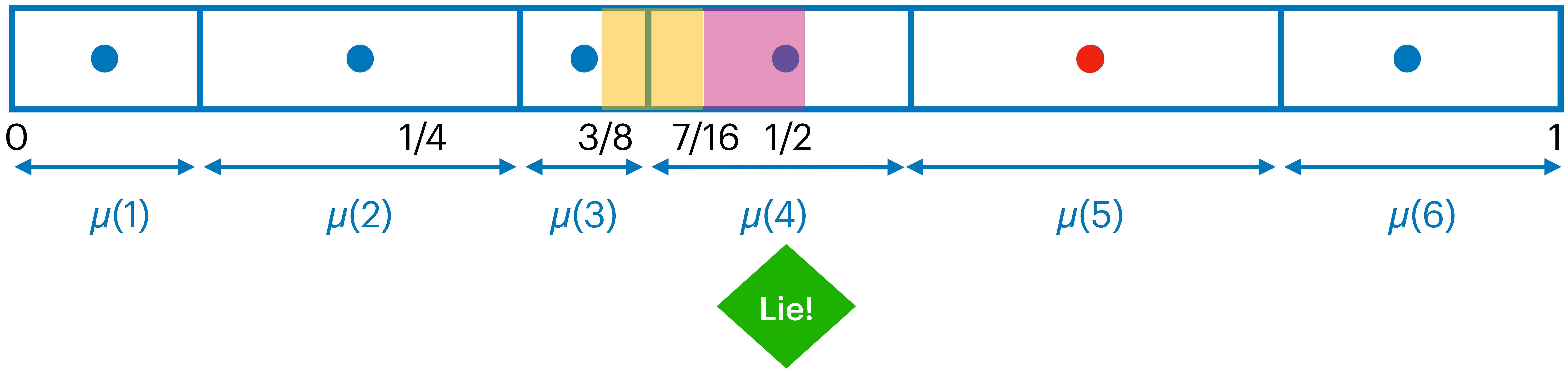

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Figure out true answer, possibly rollback

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$H(\mu) = E[\log 1/\mu]$ $H_2(\mu) = E[\log\log 1/\mu]$

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Game lasts for $\approx\log(1/\mu(x))$ rounds

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Lower bound:

- At end of game, Alice knows both x and positions where Bob lied
- Game lasts for $\approx\log(1/\mu(x))$ rounds
- Each lie position requires Alice to find $\log\log(1/\mu(x))$ more bits

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Cost incurred once per lie

Matching Huffman’s algorithm exactly

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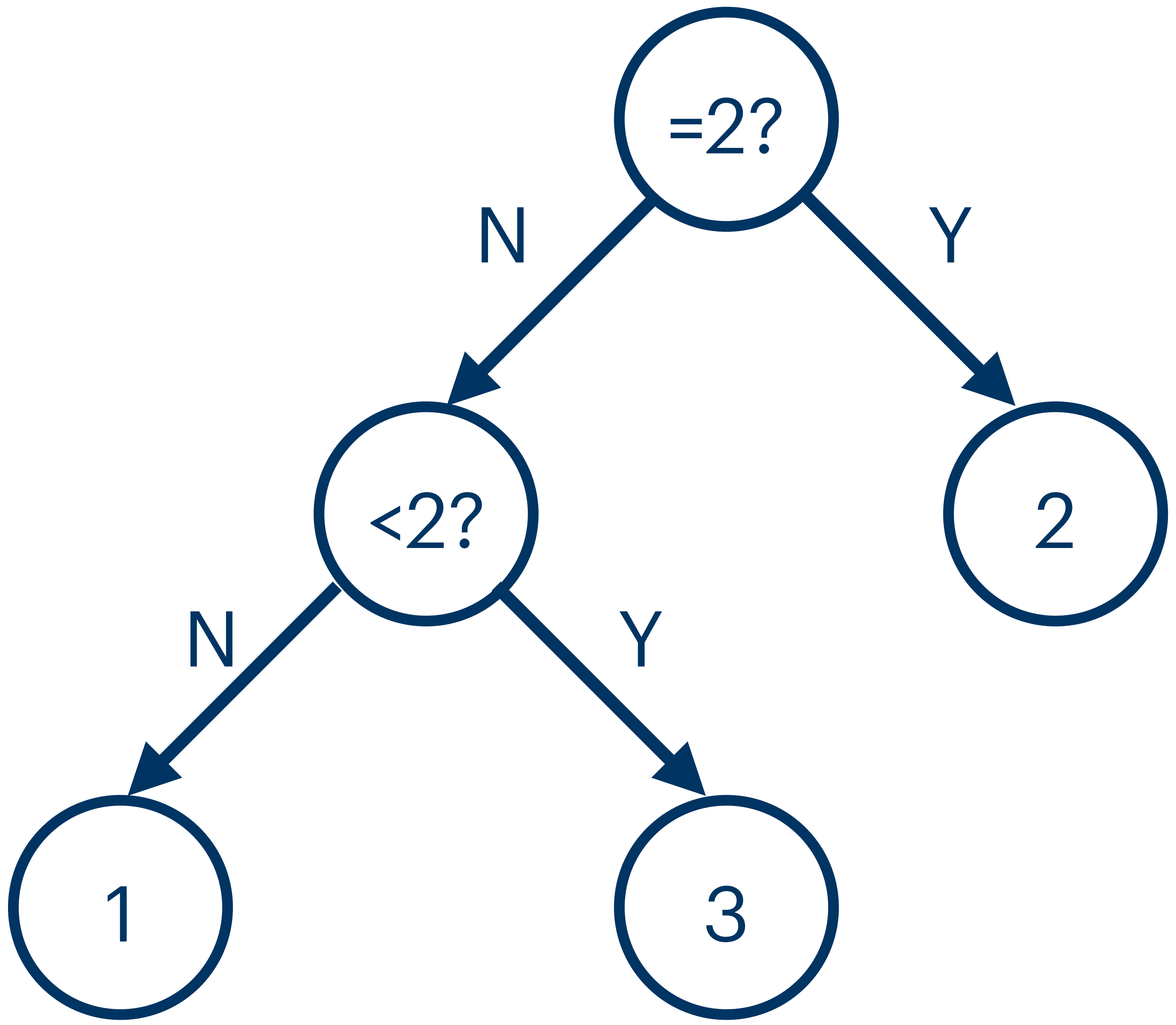

Matching Huffman's algorithm exactly

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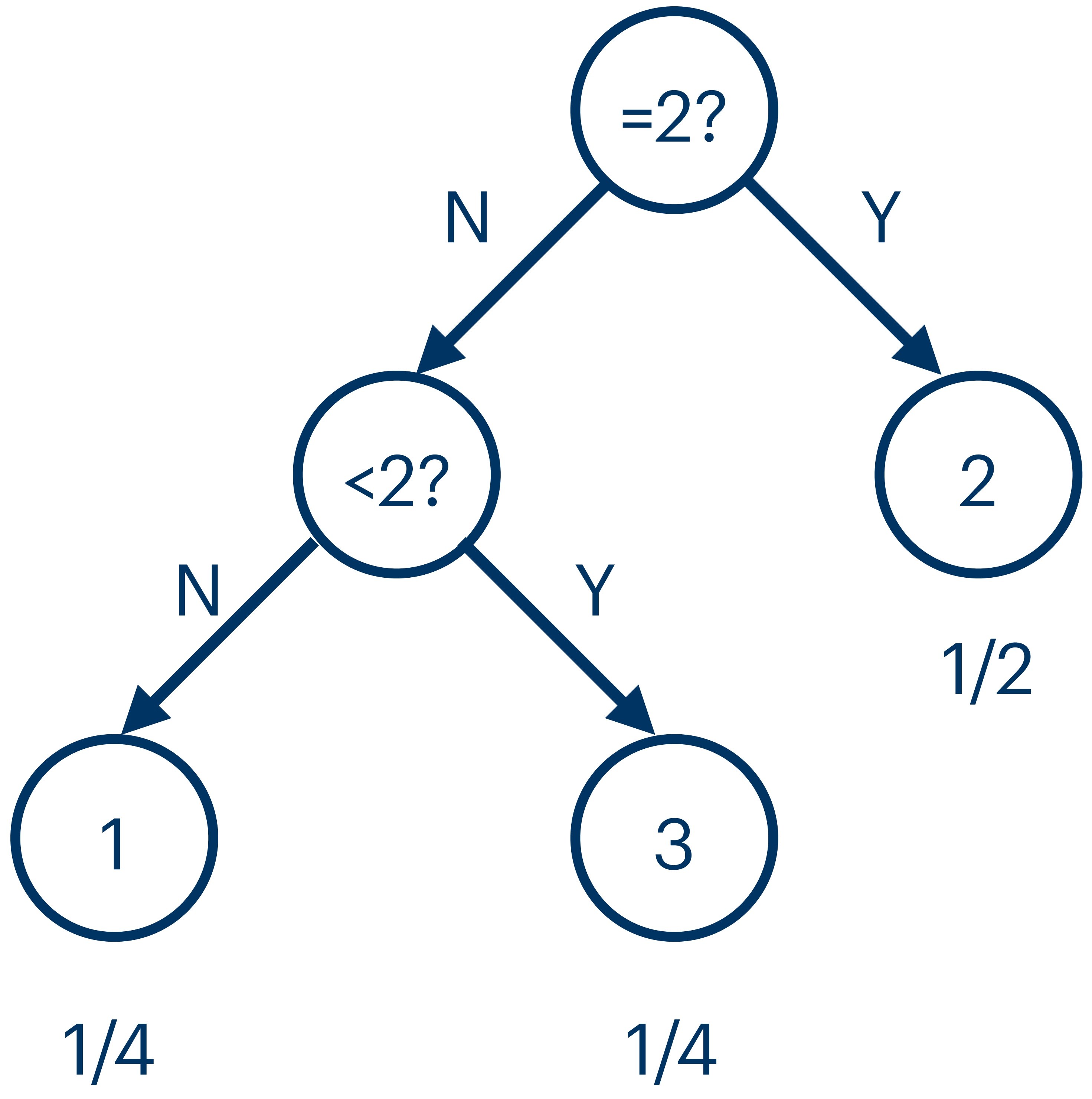

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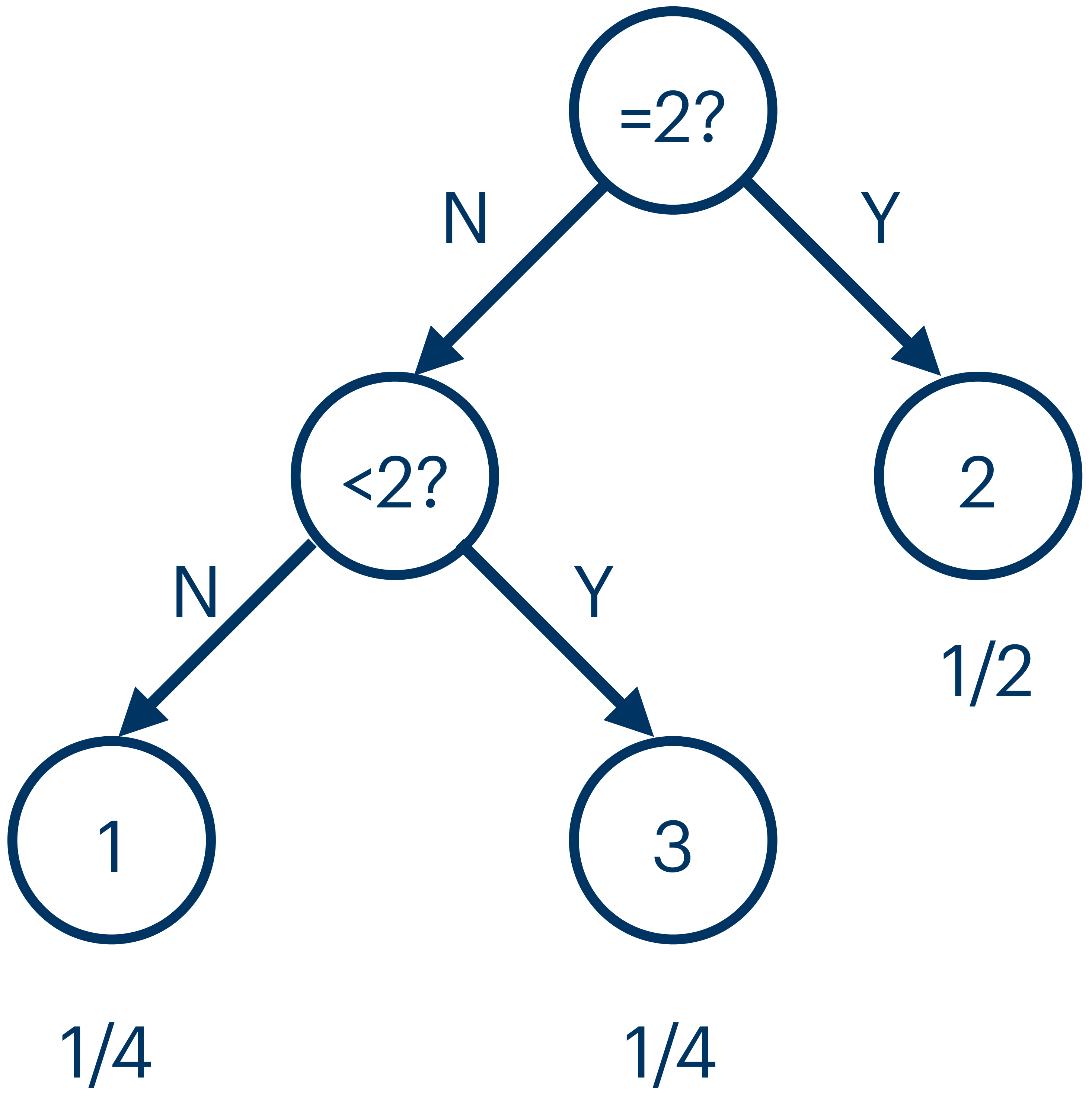

Enough to handle "dyadic" distributions

Matching Huffman's algorithm exactly

Enough to show:
Each dyadic distribution μ has a strategy
using $H(\mu)$ questions in expectation

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Equivalently:
Can always find question splitting μ evenly

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1/4	1/16	1/8	1/8	1/16	1/16	1/16	1/4
$\mu(1)$	$\mu(2)$	$\mu(3)$	$\mu(4)$	$\mu(5)$	$\mu(6)$	$\mu(7)$	$\mu(8)$

Matching Huffman's algorithm exactly

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Each dyadic distribution μ has a strategy
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Equivalently:
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Matching Huffman's algorithm exactly

Goal: Can always find question splitting μ evenly

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$\mu(1)$	$\mu(2)$	$\mu(3)$	$\mu(4)$	$\mu(5)$	$\mu(6)$	$\mu(7)$	$\mu(8)$

Example: all subsets of $\{1,\ldots,n/2\}$ + all subsets of $\{n/2+1,\ldots,n\}$

Matching Huffman's algorithm exactly

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1/4	1/16	1/8	1/8	1/16	1/16	1/16	1/4
$\mu(1)$	$\mu(2)$	$\mu(3)$	$\mu(4)$	$\mu(5)$	$\mu(6)$	$\mu(7)$	$\mu(8)$

Example: all subsets of $\{1,\ldots,n/2\}$ + all subsets of $\{n/2+1,\ldots,n\}$

Either $\mu(\{1,\ldots,n/2\}) \geq 1/2$ or $\mu(\{n/2+1,\ldots,n\}) \geq 1/2$, say the former

Matching Huffman's algorithm exactly

Goal: Can always find question splitting μ evenly

1/4	1/16	1/8	1/8	1/16	1/16	1/16	1/4
$\mu(1)$	$\mu(2)$	$\mu(3)$	$\mu(4)$	$\mu(5)$	$\mu(6)$	$\mu(7)$	$\mu(8)$

Example: all subsets of $\{1,\ldots,n/2\}$ + all subsets of $\{n/2+1,\ldots,n\}$

Either $\mu(\{1,\ldots,n/2\}) \geq 1/2$ or $\mu(\{n/2+1,\ldots,n\}) \geq 1/2$, say the former

Arrange elements in non-increasing order of probability

1/4	1/8	1/8	1/16
$\mu(1)$	$\mu(3)$	$\mu(4)$	$\mu(2)$

Matching Huffman's algorithm exactly

Goal: Can always find question splitting μ evenly

1/4	1/16	1/8	1/8	1/16	1/16	1/16	1/4
$\mu(1)$	$\mu(2)$	$\mu(3)$	$\mu(4)$	$\mu(5)$	$\mu(6)$	$\mu(7)$	$\mu(8)$

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Some prefix sums to exactly 1/2

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Optimal number of questions for Huffman + ε: $n^{O(1/\varepsilon)}$

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“<“ questions: $H(\mu)+1.086$ [Nakatsu]

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“<“ questions: H(μ)+1.086 [Nakatsu]

“<“ and “=“ questions: between H(μ)+0.501 and H(μ)+0.586

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Thank You!