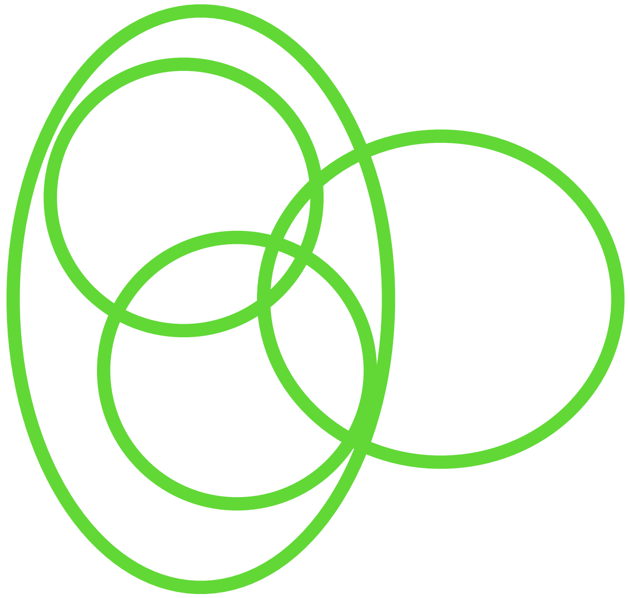
Foresight in submodular optimization

Yuval Filmus, Technion

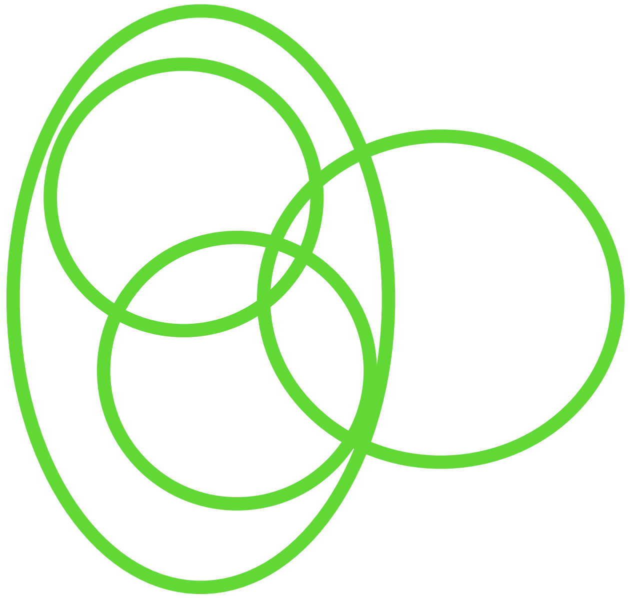
This is not Uri!

Set Cover



How many sets needed to cover entire universe?

Set Cover



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Greedy algorithm:
Repeatedly choose set covering maximum number of new elements

ln n approximation

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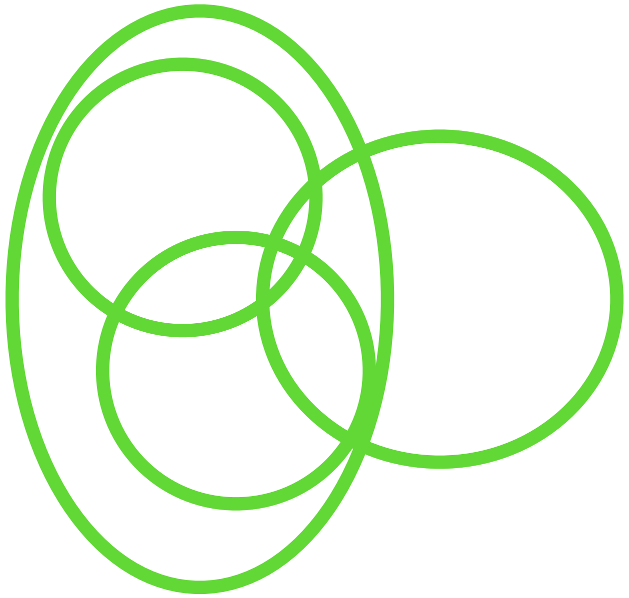

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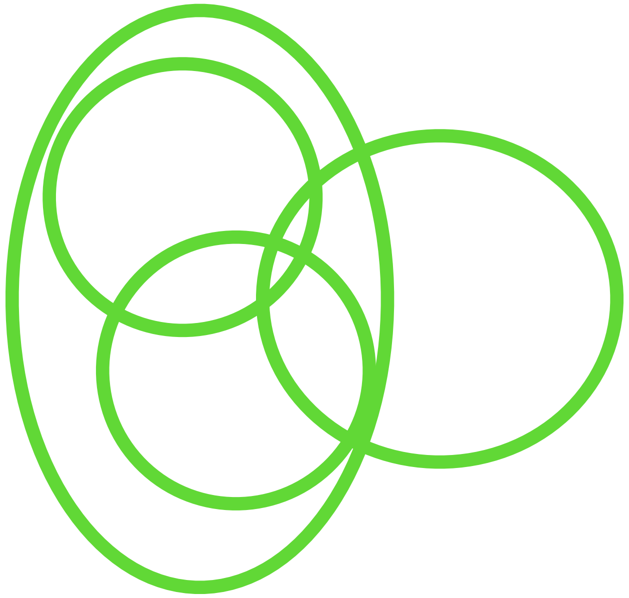
Feige (JACM'98):
Optimal unless NP$\subseteq$TIME($n^{O(\log\log n)}$)

D. Moshkovitz (2014):
Optimal unless P=NP

Max *k*-Cover



How many elements can cover using *k* sets?

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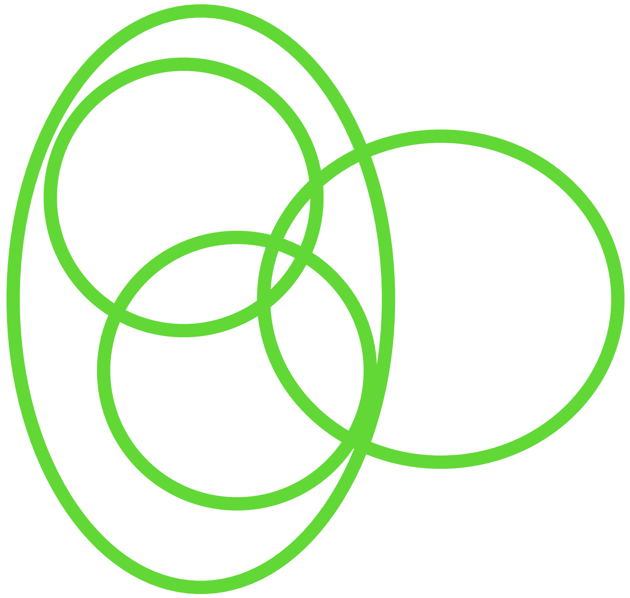

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1–1/*e* approximation
(0.632)

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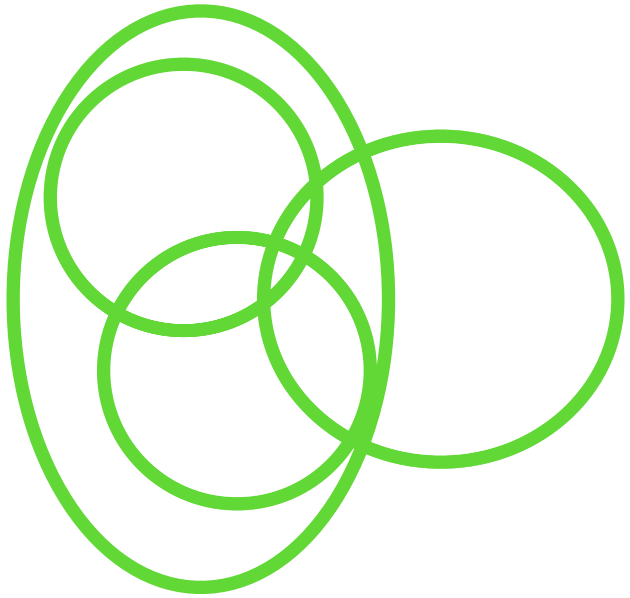

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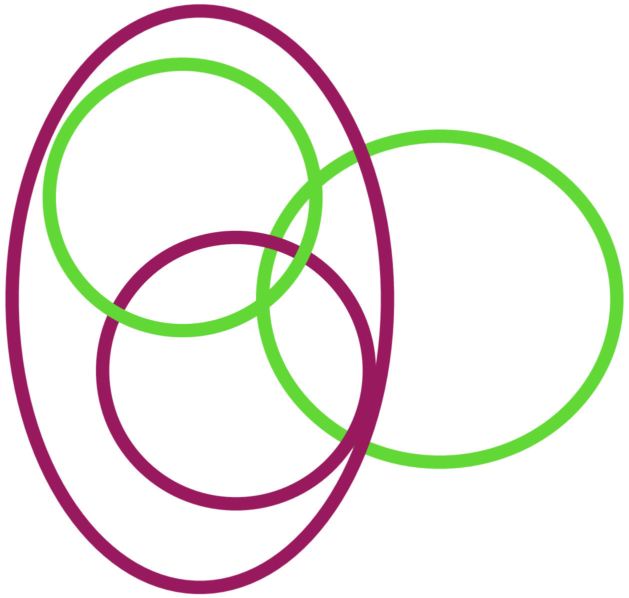
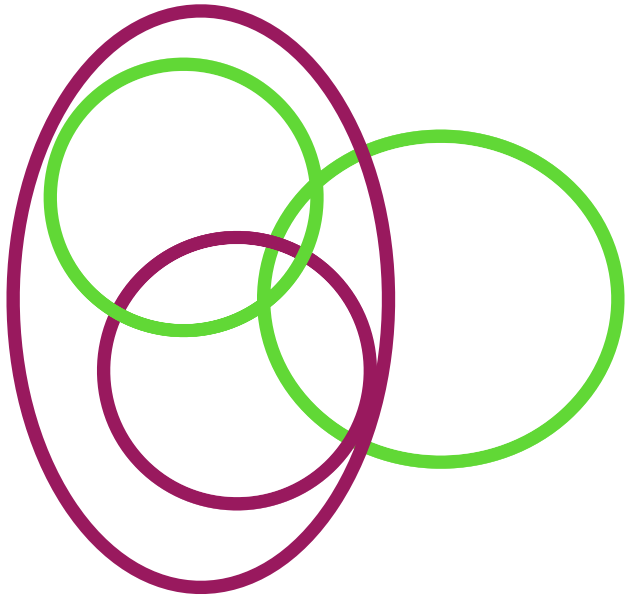


How many elements can cover using *k* sets, one of each color?

Partition Max *k*-Cover



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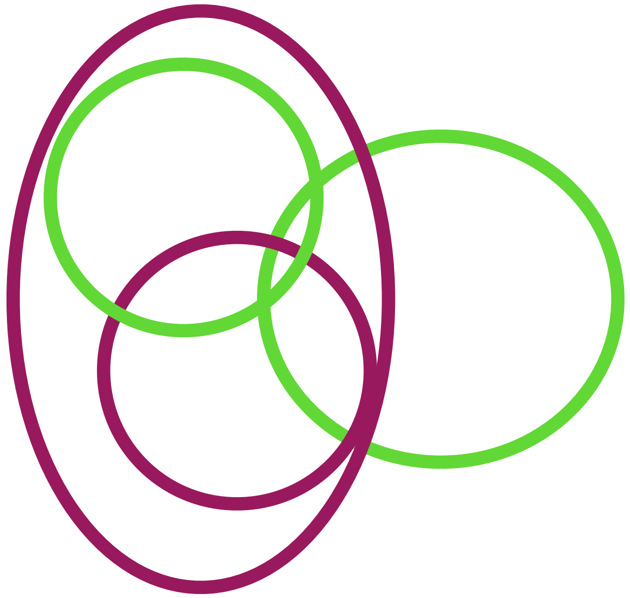

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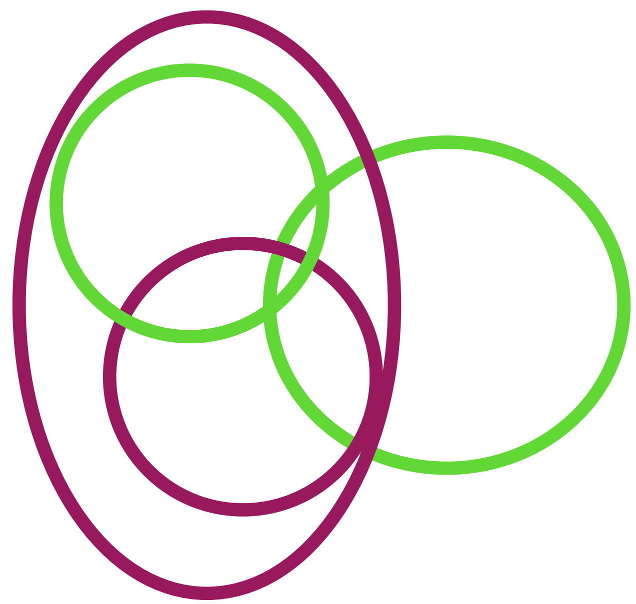
1/2 approximation

Generalizes Max SAT:

- Color: variable
- Set: truth assignment
- Element: clause

Partition Max *k*-Cover



How many elements can cover using *k* sets, one of each color?

Random order greedy:
Choose random order of colors.
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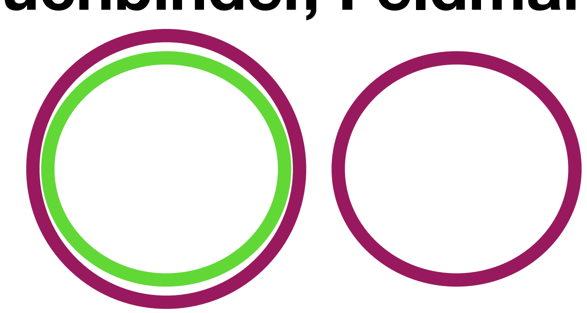
≥0.5096 approximation
[Buchbinder, Feldman, F, Garg'19]



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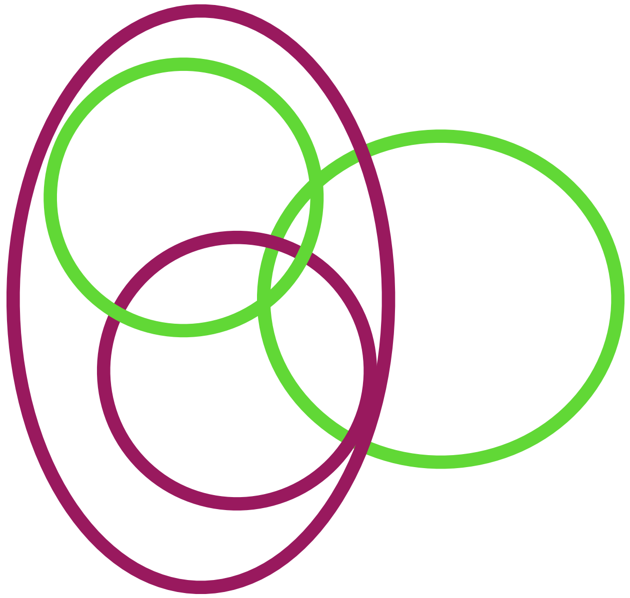

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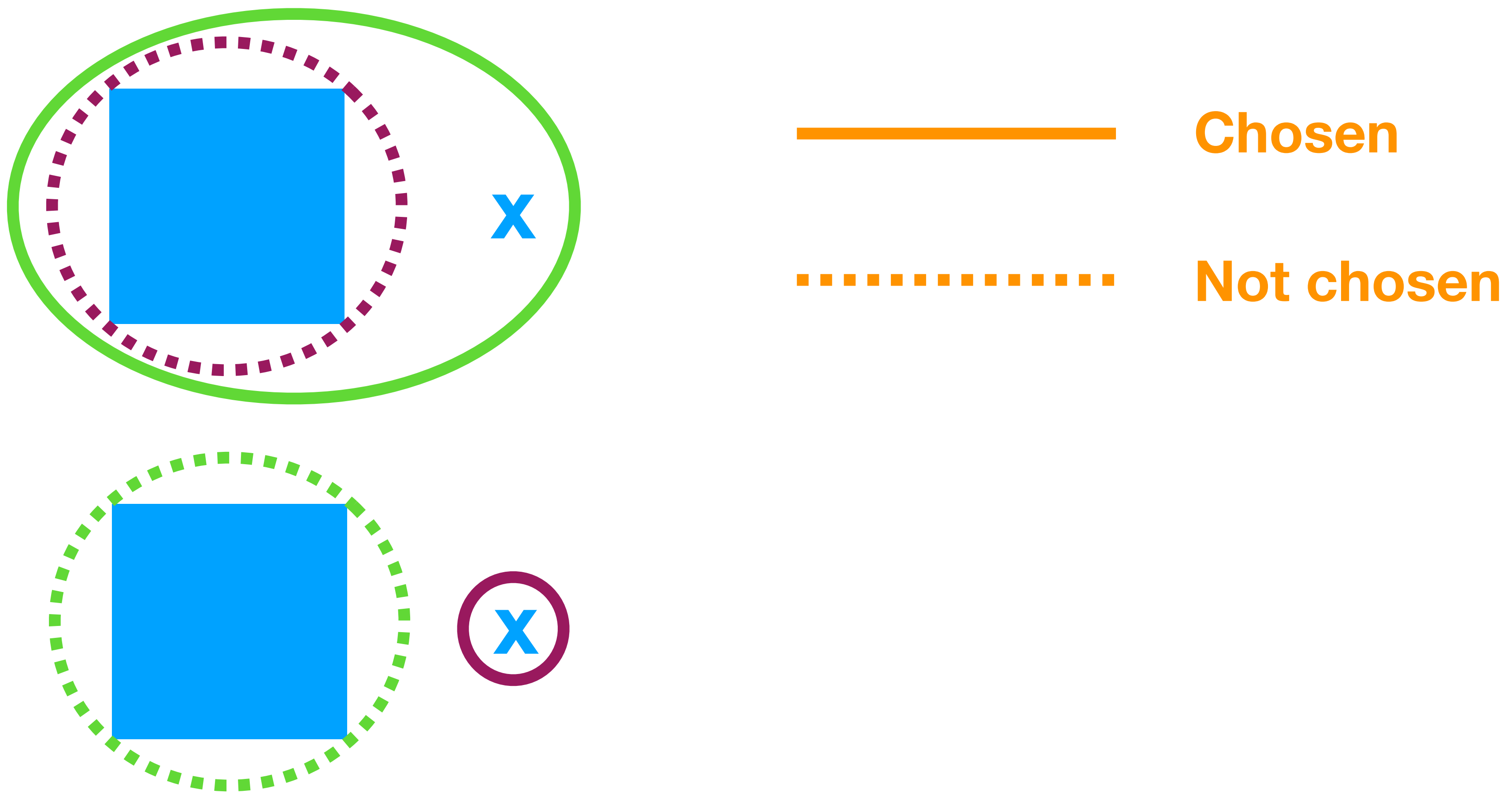
Srinivasan '01,
Ageev, Sviridenko '04:
LP relaxation

Calinescu, Chekuri, Pál, Vondrák '09:
Continuous greedy

1–1/*e* approximation

Not combinatorial!

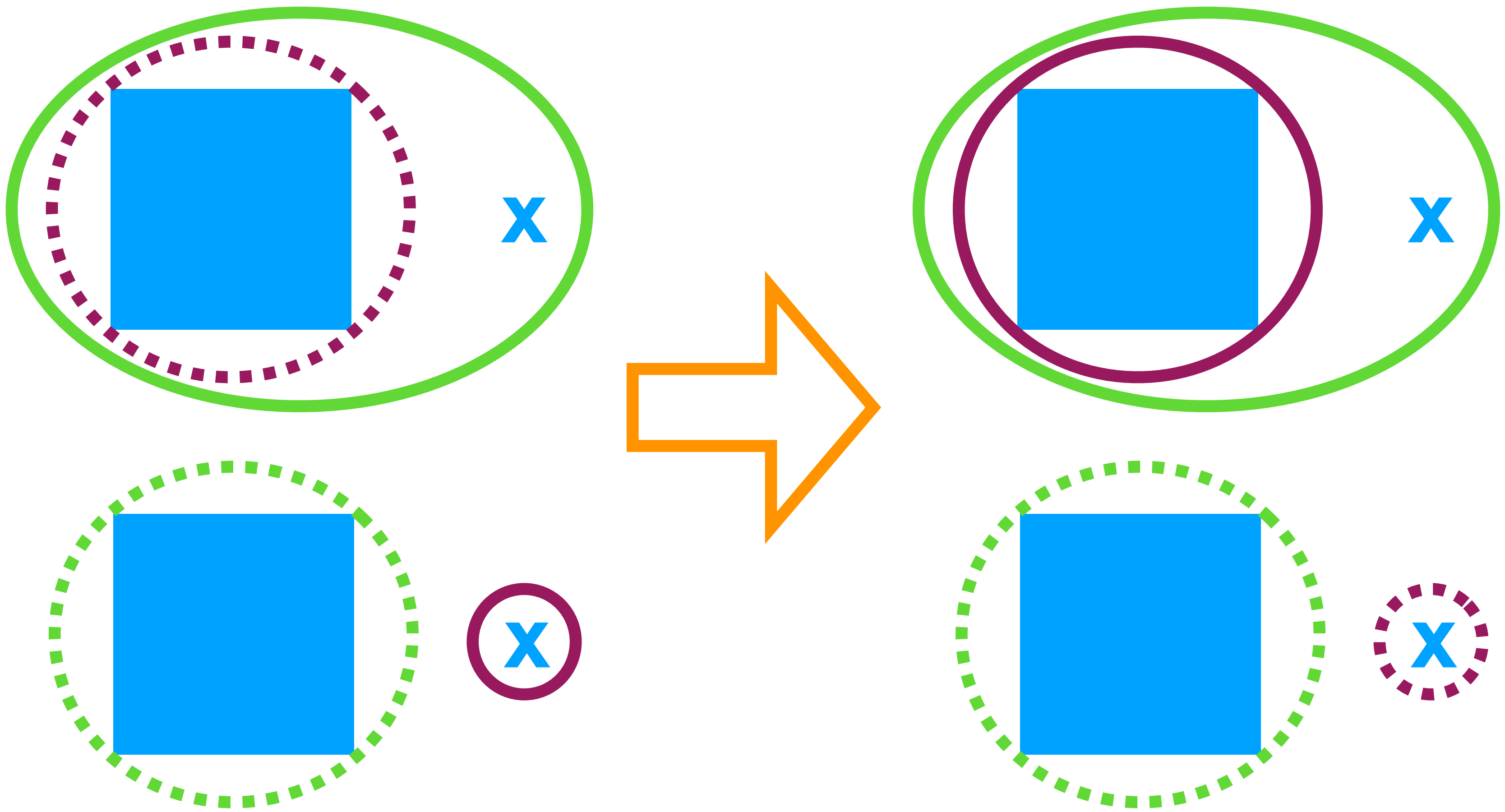
Local search?



Locally optimal!

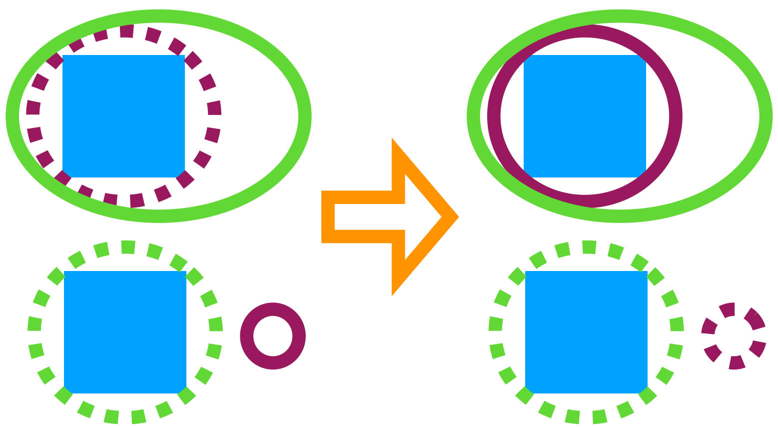
1/2 approximation

Local search?



Loss: Cover fewer elements
Gain: Upper square covered twice

Local search!



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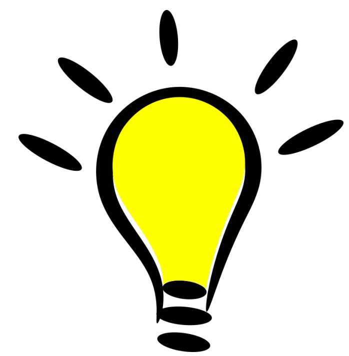
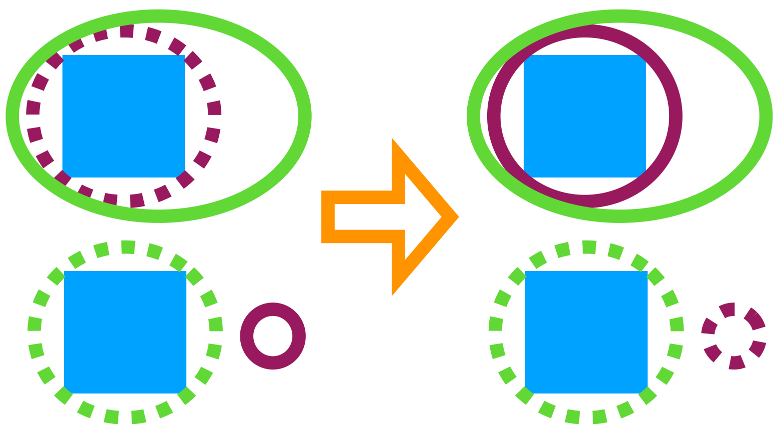


Idea:
Give more weight to elements covered multiple times

Local search!



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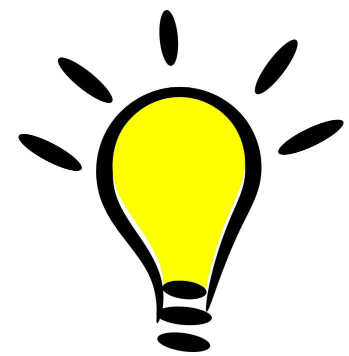

Idea:
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F, Ward (2012):
Optimal choice of weights
➜ 1–1/*e* approximation!

Optimal weights found by solving infinite LP

Multiplicity	Weight
0	0
1	1
2	1.418
3	1.672
4	1.852
5	1.991
k	$\approx C \log k$

$$\alpha_{k+1} = (k+1)\alpha_k - k\alpha_{k-1} - \frac{1}{e-1}$$

Analyzing local search

Setup:

- **Local optimum** $S_1,\ldots,S_k$
- **Global optimum** $O_1,\ldots,O_k$

Switching S_i and O_i doesn't improve objective function $cost \Rightarrow$

$$\sum_{i=1}^{k} cost(S_1, \ldots, O_i, \ldots, S_k) \leq k \cdot cost(S_1, \ldots, S_k) \qquad \text{(local optimality)}$$

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$N(a,b,c)$ – # elements appearing:
a times in $S_1,\dots,S_k$
b times in $O_1,\dots,O_k$
c times in $S_1 \cap O_1,\dots,S_k \cap O_k$

Given weights, can write LP for approximation ratio in variables $N(a,b,c)$:

min $|S_1 \cup \cdots \cup S_k|$ *s.t.* $|O_1 \cup \cdots \cup O_k|=1$ and "local optimality"

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Can find optimal weights using dual LP

Matroids

Setup:

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Brualdi's lemma:
If $S_1,\ldots,S_k$ and $O_1,\ldots,O_k$ are two bases in an arbitrary matroid, can rearrange indices so that $S_1,\ldots,O_i,\ldots,S_k$ is a basis for all i

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Conclusion: algorithm works for arbitrary matroids

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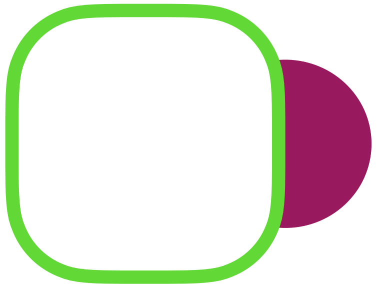

$f(A+x) \geq f(A)$ **Monotonicity**

Submodular functions

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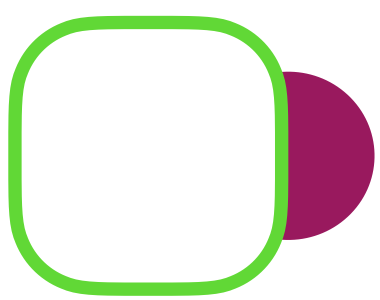
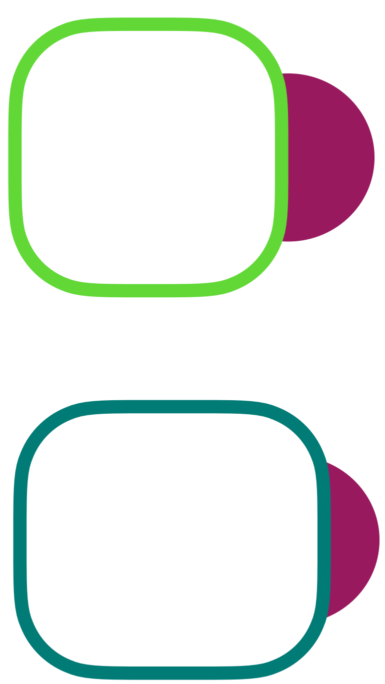
	$f(\emptyset) = 0$	**Normalization**
	$f(A+x) \geq f(A)$	**Monotonicity**
	**If** $A \subseteq B$ **then** $f(A+x) - f(A) \geq f(B+x) - f(B)$	**Submodularity**

Submodular functions

Alternative definitions:

- **Normalization:** $f(\varnothing)=0$
- **Monotonicity:** $f(A+x)-f(A)\geq 0$ $\qquad \partial_x f \geq 0$
- **Submodularity:** $f(A+x+y)-f(A+x)-f(A+y)+f(A)\leq 0$ $\qquad \partial_x \partial_y f \leq 0$

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Coverage functions characterized by:

$$\partial_{x_1}\cdots\partial_{x_k} f \begin{cases} \geq 0 & \text{if } k \text{ odd} \\ \leq 0 & \text{if } k \text{ even} \end{cases}$$

Many algorithms for coverage functions only need condition on first two derivatives

Submodular functions

"Max *k*-cover":
Given monotone submodular *f*,
maximize *f*(*A*) over |*A*|=*k*

Greedy algorithm:
Repeatedly add elements
maximizing value of *f*

1–1/*e* approximation

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Given monotone submodular *f*,
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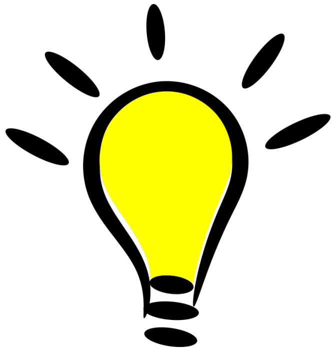
Can we generalize our local search algorithm?

Submodular functions



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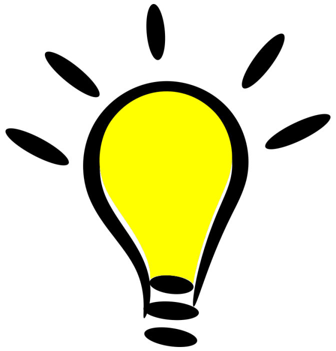
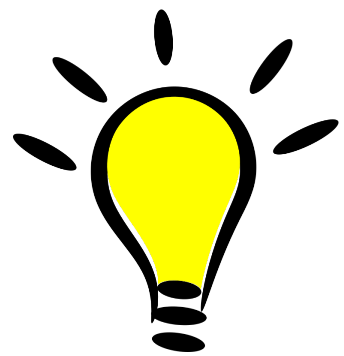


Idea:
Formulate $cost$(*A*) in terms of *f*(*B*) for $B \subseteq A$

Let $f_p(A)$=E[*f*(*B*)], where *B* is chosen by sampling each element of *A* w.p. *p*

De Finetti's theorem: $cost$ is mixture of f_p

Submodular functions



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$$cost(A) = \frac{1}{e-1}\int_0^1 \frac{e^p}{p} f_p(A)\,\mathrm{d}p \qquad \partial_x cost(A) = \int_0^1 \frac{e^p}{e-1} (\partial_x f)_p(A)\,\mathrm{d}p$$

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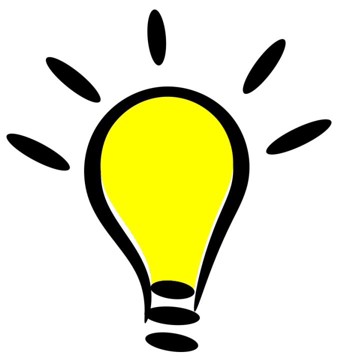

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F, Ward (2012): 1–1/*e* approximation for any monotone submodular *f*!

Continuous greedy

Vondrák (2008)

Competing algorithm for submodular max k-cover:

Maintain feasible solution A
At time $p \in [0,1]$, for each color i, at rate $1/p$:
Update color i with x maximizing $(\partial_x f)_p(A)$

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Exact connection between the two algorithms is still a mystery!

Open problems

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- Tight analysis of random order greedy
- Random order greedy with foresight?
  (“Secretary” variant of partition max k-cover)

Deterministic algorithm?

Our algorithm is randomized since f_p can only be computed by sampling
Continuous greedy algorithm is randomized for similar reasons

Best known deterministic algorithm:
Residual greedy [Buchbinder, Feldman, Garg '19], 0.5008 approximation

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Buchbinder, Feldman ’16: deterministic 1/2 approximation

Happy birthday, Uri!

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