Friedgut–Kalai–Naor for S_n

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Introduction

Suppose $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is linear:

$$f(x_1, \ldots, x_n) = c_0 + \sum_i c_i x_i.$$

Then $f \in \{\pm 1, \pm x_i\}$.

Application: uniqueness for μ_p-EKR.

Friedgut–Kalai–Naor

Suppose $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is almost linear:

$$f(x_1,\ldots,x_n) \approx c_0 + \sum_i c_i x_i.$$

Then $f \approx \{\pm 1, \pm x_i\}$.

Application: stability for μ_p-EKR.

Friedgut–Kalai–Naor

Suppose $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is almost linear:

$$\mathbb{E}[(f(x_1,\ldots,x_n) - (c_0 + \sum_i c_i x_i))^2] = \epsilon.$$

Then $\mathbb{E}[(f-g)^2] = O(\epsilon)$ for some $g \in \{\pm 1, \pm x_i\}$.

Application: stability for μ_p-EKR.

Benabbas–Friedgut–Pilpel

Suppose $f\colon S_n \to \{0,1\}$ is linear:

$$f = \sum_{i,j} c_{ij} T_{ij}$$

where T_{ij} is the set of permutations sending i to j.
Then f is a disjoint union of T_{ij}'s, i.e. a dictator:
$f(\pi)$ depends only on $\pi(i)$ or $\pi^{-1}(j)$.

Application: uniqueness for S_n-EKR.

Conjecture

Suppose $f\colon S_n \to \{0,1\}$ is almost linear:

$$f \approx \sum_{i,j} c_{ij} T_{ij}.$$

Then $f \approx$ union of T_{ij}'s.

Application: stability for S_n-EKR.

Conjecture

Suppose $f\colon S_n \to \{0,1\}$ is almost linear:

$$\mathbb{E}[(f - \sum_{i,j} c_{ij} T_{ij}))^2] = \epsilon.$$

Then $\mathbb{E}[(f-g)^2] = O(\epsilon)$ for some $g =$ union of T_{ij}'s.

Application: stability for S_n-EKR.

EFF1 (sparse functions)

Suppose $f\colon S_n \to \{0,1\}$ is almost linear:

$$\mathbb{E}[(f - \sum_{i,j} c_{ij} T_{ij})^2] = \epsilon\, \mathbb{E}[f].$$

Then for some $g =$ union of $c = \lceil n\,\mathbb{E}[f] \rfloor$ T_{ij}'s,

$$\mathbb{E}[(f - g)^2] = O(\sqrt{\epsilon} + \frac{1}{n}) c\, \mathbb{E}[f].$$

Cannot guarantee cosets to be disjoint!
(Think of $T_{11} + T_{22} - T_{12,12}$.)

EFF2 (balanced functions)

Suppose $f\colon S_n \to \{0,1\}$ is almost linear:

$$\mathbb{E}[(f - \sum_{i,j} c_{ij} T_{ij})^2] = \epsilon.$$

Then for some *dictator* $g =$ *disjoint* union of $\lceil n\,\mathbb{E}[f] \rfloor$ T_{ij}'s,

$$\mathbb{E}[(f-g)^2] = O(\frac{1}{\eta}(\epsilon^{1/7} + \frac{1}{n^{1/3}})),$$

where $\eta = \min(\mathbb{E} f, 1 - \mathbb{E} f)$.

EFF2 — Setup

- f is a ± 1 function.
- f_1 = projection of f into span of T_{ij}'s.
- $\mathbb{E}[(f - f_1)^2] = \epsilon$.
- For simplicity, assume $\mathbb{E} f = 0$.

Canonical representation of f_1:

$$f_1 = \sum_{i,j} a_{ij} T_{ij}, \quad \text{where } a_{ij} = (n-1)\langle f, T_{ij} \rangle,$$

$$f_1(\pi) = \sum_{i} a_{i\pi(i)}.$$

Matrix representation

Matrix (a_{ij}) for $f = T_{11} \cup \cdots \cup T_{1(n/2)}$:

$$\begin{pmatrix} \overbrace{1-\frac{1}{n} & \cdots & 1-\frac{1}{n}}^{\frac{n}{2}} & \frac{1}{n}-1 & \cdots & \frac{1}{n}-1 \\ -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots & \vdots & & \vdots \\ -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

$f_1(\pi) = \sum_i a_{i\pi(i)}$ (generalized diagonal)

Step 1

Choose $X, Y \subseteq [n]$ randomly of same size.

$$g \triangleq f_1|_{\pi:\pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}}),$$
$$g_1(\pi_1) = \sum_{i \in X} a_{i\pi_1(i)}, \qquad g_2(\pi_2) = \sum_{i \in \overline{X}} a_{i\pi_2(i)}.$$

Whp over choice of X, Y:

- g is *close* to ± 1
- $\mathbb{E}\, g_1, \mathbb{E}\, g_2$ *close* to 0

(X, Y are *reasonable*.)

Step 1 (cont.)

Whp over choice of X, Y:

- g is *close* to ± 1
- $\mathbb{E}\, g_1, \mathbb{E}\, g_2$ *close* to 0

Inputs to g_1, g_2 are *independent*.
How can $g = g_1 + g_2$ be close to ± 1?

- One of g_1, g_2 must be almost constant $-X$, other close to $X \pm 1$.
- Since $\mathbb{E}\, g_1 \approx \mathbb{E}\, g_2 \approx 0$, $X \approx 0$.
- So one of g_1, g_2 is almost zero, other close to ± 1.

Step 2

Whp over $\pi \in S_n$:

- Most $X, \pi(X)$ are *reasonable*.
- So for most X,

$$\sum_{i \in X} a_{i\pi(i)} \approx 0, \quad \sum_{i \in \overline{X}} a_{i\pi(i)} \approx \pm 1$$

  or vice versa.
- This can only happen if one $a_{i\pi(i)}$ is *large* ($\approx \pm 1$) and all other $a_{i\pi(i)}$ are *small* (≈ 0).

Conclusion: almost all *generalized diagonals* in (a_{ij}) have exactly one large entry.

Step 3

If all generalized diagonals in (a_{ij}) have one large entry then

- All entries in some row or column are large.

We prove a stability version of this result:

- In some row/column, almost all entries are large.
- Every row/column sums to zero, so approximately half are $+1$, half are -1.
- Union of $+1$ cosets approximates f.

□

EFF1 — Setup

- f is a $\{0,1\}$ function.
- f_1 = projection of f into span of T_{ij}'s.
- $\mathbb{E}[(f - f_1)^2] = \epsilon$.
- For simplicity, assume $\mathbb{E} f = 1/n$.

Canonical representation of f_1:

$$h = \sum_{i,j} b_{ij} T_{ij}, \quad \text{where } b_{ij} = n\langle f, T_{ij}\rangle - \frac{1}{n}.$$

$$h = \frac{n}{n-1} f_1 - \frac{1}{n-1} \approx f_1.$$

Note $\mathbb{E}\, h = 0$, $h \geq -1/(n-1)$.

Matrix representation

Matrix (b_{ij}) for $f = T_{11}$:

$$\begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{1}{n(n-1)} & \cdots & \frac{1}{n(n-1)} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{n} & \frac{1}{n(n-1)} & \cdots & \frac{1}{n(n-1)} \end{pmatrix}$$

- Rows and columns sum to 0.
- $\sum_{i,j} b_{ij}^2 \approx \sum_{i,j} b_{ij}^3 \approx 1.$

Moments of the matrix

Moments:

$$\mathbb{E}[h^2] = \frac{1}{n-1} \sum_{i,j} b_{ij}^2,$$

$$\mathbb{E}[h^3] = \frac{n}{(n-1)(n-2)} \sum_{i,j} b_{ij}^3.$$

Estimating the moments:

- $f \approx f_1 \implies \mathbb{E}[h^2] \approx \frac{1}{n}$.
- $\mathbb{E}[h^3] \gtrapprox \frac{1}{n}$ follows from:
  1. $\mathbb{E}\, h = 0$,
  2. $h + 1 \geq 0$,
  3. $h + 1$ is L_2-close to $f + 1$,
  4. $f + 1$ is 1 w.p. $1 - 1/n$ and 2 w.p. $1/n$.

Finishing the proof

- $\mathbb{E}[h^2] \approx \frac{1}{n}$ and $\mathbb{E}[h^3] \gtrapprox \frac{1}{n}$.
- So $\sum_{i,j} b_{ij}^2 \approx 1$ and $\sum_{i,j} b_{ij}^3 \gtrapprox 1$.
- So $\sum_{i,j} b_{ij}^2(1 - b_{ij}) \approx 0$.
- So b_{ij}'s are close to $\{0, 1\}$.
- Since $\sum_{i,j} b_{ij}^2 \approx 1$, exactly one b_{ij} is close to 1.
- So $f \approx T_{ij}$.

□

Johnson scheme

Suppose $f\colon \binom{[n]}{k} \to \{-1,1\}$ is linear:

$$f(x_1,\ldots,x_n) = \sum_i c_i x_i.$$

Then $f \in \{\pm 1, \pm x_i\}$.

Application: uniqueness for EKR.

FKN for Johnson scheme

Suppose $f\colon \binom{[n]}{k} \to \{\pm 1\}$ is almost linear:

$$f(x_1, \ldots, x_n) \approx \sum_i c_i x_i.$$

Then $f \approx \{\pm 1, \pm x_i\}$.

Application: stability for EKR.

FKN for Johnson scheme

Suppose $f\colon \binom{[n]}{k} \to \{\pm 1\}$ is almost linear:

$$\mathbb{E}[(f(x_1,\ldots,x_n) - (\sum_i c_i x_i))^2] = \epsilon \leq c\frac{k^2}{n^2}.$$

Then $\mathbb{E}[(f-g)^2] = O(\epsilon)$ for some $g \in \{\pm 1, \pm x_i\}$.

Application: stability for EKR.