Structure theorems for almost low degree functions

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   - Almost linear functions on S_n (sparse case)
   - Almost linear functions on S_n (balanced case)

Section 1

Theorems

Linear functions

Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ is linear:

$$f(x_1,\ldots,x_n) = c_0 + \sum_{i=1}^n c_i x_i.$$

Linear functions

Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ is linear:

$$f(x_1,\ldots,x_n) = c_0 + \sum_{i=1}^n c_i x_i.$$

Theorem: f depends on at most one coordinate.

Almost linear functions

Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ is almost linear:

$$\mathbb{E}\left[\left(c_0 + \sum_{i=1}^n c_i x_i - f\right)^2\right] = \epsilon.$$

Distribution over $\{-1,1\}^n$: uniform or μ_p.

$$\mu_p(x_1,\ldots,x_n) = p^{\#\{i:x_i=-1\}|}(1-p)^{\#\{i:x_i=1\}}.$$

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Theorem: f is $O(\epsilon)$-close to a linear Boolean function.
(Friedgut–Kalai–Naor, 2002)

(Almost) low-degree functions

If $f\colon \{-1,1\}^n \to \{-1,1\}$ has degree d then
f depends on $\leq d2^d$ variables.
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If $f\colon \{-1,1\}^n \to \{-1,1\}$ is ϵ-close to a function of degree d then
f is $O(\epsilon)$-close to a *Boolean* function of degree d.
(Kindler–Safra, 2002)

Functions on the slice

The slice is $\binom{[n]}{k}$.

Usually assume $\delta \leq \frac{k}{n} \leq 1 - \delta$.

Can identify the slice with

$$\left\{(x_1, \ldots, x_n) \in \{\pm 1\}^n : \sum_{i=1}^n x_i = 2k - n\right\}.$$

Linear functions

Suppose $f\colon \binom{[n]}{k} \to \{-1,1\}$ is linear:

$$f(x_1,\ldots,x_n) = \sum_{i=1}^n c_i x_i.$$

Theorem: f depends on at most one coordinate.

Almost linear functions

Suppose $f\colon \binom{[n]}{k} \to \{-1,1\}$ is almost linear:

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} c_i x_i - f\right)^2\right] = \epsilon.$$

Uniform distribution on $\binom{[n]}{k}$.

Theorem: f is $O(\epsilon)$-close to a linear Boolean function.
(F. et al., 2013+)

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If $f\colon \binom{[n]}{k} \to \{-1,1\}$ has degree d then f depends on $\exp(d)$ variables.
(F. et al., 2013+)

Conjecture: If $f\colon \binom{[n]}{k} \to \{-1,1\}$ is ϵ-close to a function of degree d then f is $O(\epsilon)$-close to a *Boolean* function of degree d.

Functions on the slice

The symmetric group is S_n.

Can identify S_n with permutation matrices $X = (x_{ij})_{i,j=1}^n$.
Each entry is $0/1$, each row and each column sums to 1.

Linear functions

Suppose $f\colon S_n \to \{0,1\}$ is linear:

$$f(X) = \sum_{i,j=1}^{n} c_{ij} x_{ij}.$$

Theorem: f depends on at most one row or one column.

$$f(\pi) = [\![\pi(i) \in J]\!]$$

or

$$f(\pi) = [\![\pi^{-1}(j) \in I]\!]$$

(Ellis, Friedgut and Pilpel, 2011)

Almost linear functions

Suppose $f\colon S_n \to \{0,1\}$ is almost linear:

$$\mathbb{E}\left[\left(\sum_{i,j=1}^{n} c_{ij} x_i - f\right)^2\right] = \epsilon.$$

Uniform distribution on S_n.

Theorem: If f is balanced, f is $O(\epsilon^{1/7})$-close to a linear Boolean function.

Theorem: If f is sparse, f is $O(\epsilon^{1/2})$-close to a function of the form

$$\max(x_{i_1 j_1}, \ldots, x_{i_r j_r}),$$

i.e., characteristic function of a union of double cosets

$$T_{ij} = \{\pi \in S_n : \pi(i) = j\}.$$

Sparse means $\mathbb{E}[f] = c/n$ for small c.

(Ellis, F., Friedgut, 2014)

(Almost) low-degree functions

If $f : S_n \to \{0, 1\}$ has degree d then
f can be written as a sum of disjoint monomials of degree d.
I.e., f is characteristic function of disjoint sum of double d-cosets

$$T_{i_1 j_1} \cap \cdots \cap T_{i_d j_d}.$$

(Ellis, Friedgut and Pilpel, 2011)

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Theorem: If f is sparse, f is $O(\epsilon^{1/2})$-close to the characteristic function of a union of double d-cosets.
Sparse means $\mathbb{E}[f] = c/n^d$ for small c.
(Ellis, F., Friedgut, 2014)

Section 2

Applications

Erdős–Ko–Rado theorems

Erdős–Ko–Rado theorem (1938/1961):
If $k < n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Equality only for $\mathcal{F} = \{S \in \binom{[n]}{k} : i \in S\}$ ("star").

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If $p < 1/2$ and $\mathcal{F} \in 2^{[n]}$ is intersecting, then $\mu_p(\mathcal{F}) \leq p$.
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(Friedgut, 2008)

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Equality only for $\mathcal{F} = \{S \in 2^{[n]} : i \in S\}$.
(Friedgut, 2008)

If $\mathcal{F} \subseteq S_n$ is intersecting, then $|\mathcal{F}| \leq (n-1)!$.
Equality only for $\mathcal{F} = \{\pi \in S_n : \pi(i) = j\}$.
(Deza–Frankl, 1977); (Cameron–Ku, 2003)

Stability versions

If $k < n/2$, $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting,
and $|\mathcal{F}| \approx \binom{n-1}{k-1}$, then $\mathcal{F}$ is close to a star.
(Frankl, 1987)

If $p < 1/2$, $\mathcal{F} \in 2^{[n]}$ is intersecting,
and $\mu_p(\mathcal{F}) \approx p$, then $\mathcal{F}$ is close to a star.
(Friedgut, 2008)

If $\mathcal{F} \subseteq S_n$ is intersecting
and $|\mathcal{F}| \approx (n-1)!$, then $\mathcal{F}$ is close to a star.
(Ellis, 2009)

Stability and structure theorems

Stability theorems follow from structure theorems.
Example: Intersecting families in $2^{[n]}$.
Let f be characteristic function of an intersecting family.
Friedgut constructs a quadratic form Q such that $\langle f, Qf \rangle = 0$.
Spectral decomposition of Q implies

$$\sum_{S \subseteq [n]} \left(-\frac{p}{1-p}\right)^{|S|} \hat{f}(S)^2 = 0.$$

Also know $\hat{f}(\emptyset) = \sum_S \hat{f}(S)^2 = \mu_p(\mathcal{F})$.

Stability and structure theorems

For characteristic function f of intersecting family $\mathcal{F}$:

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Also know $\hat{f}(\emptyset) = \sum_S \hat{f}(S)^2 = \mu_p(\mathcal{F})$.

- $\mu_p(\mathcal{F}) \leq p$.
- If $\mu_p(\mathcal{F}) = p$ then $\hat{f}$ is supported on first two levels.
- If $\mu_p(\mathcal{F}) \approx p$ then $\hat{f}$ is concentrated on first two levels.

Friedgut–Kalai–Naor gives stability.

Multiple intersections

If $n \geq (t+1)(k-t-1)$ and $\mathcal{F}$ is t-intersecting, then

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Equality only for $\mathcal{F} = \{S \in \binom{[n]}{k} : i_1, \ldots, i_t \in S\}$ ("t-star").

(Frankl, 1984)

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If $p < 1/(t+1)$ and $\mathcal{F} \in 2^{[n]}$ is t-intersecting, then $\mu_p(\mathcal{F}) \leq p^t$.
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Equality only for $\mathcal{F} = \{S \in 2^{[n]} : i_1, \ldots, i_t \in S\}$.
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If $n \geq C_t$ and $\mathcal{F} \subseteq S_n$ is t-intersecting, then $|\mathcal{F}| \leq (n-t)!$.
Equality only for $\mathcal{F} = \{\pi \in S_n : \pi(i_1) = j_1, \ldots, \pi(i_t) = j_t\}$.
C_t should be equal to $2t+1$.
(Ellis, Friedgut and Pilpel, 2011)

Multiple intersections: stability

Conjecture: if $n \geq (t+1)(k-t-1)$, $\mathcal{F}$ is t-intersecting, and $|\mathcal{F}| \approx \binom{n-t}{k-t}$, then $\mathcal{F}$ is close to a t-star.

If $p < 1/(t+1)$, $\mathcal{F} \in 2^{[n]}$ is t-intersecting, and $\mu_p(\mathcal{F}) \approx p^t$, then $\mathcal{F}$ is close to a t-star.
(Friedgut, 2008), proof uses (Kindler–Safra, 2002)

If $n \geq C_t$, $\mathcal{F} \subseteq S_n$ is t-intersecting, and $|\mathcal{F}| \approx (n-t)!$, then $\mathcal{F}$ is close to a t-star.
(Ellis, 2011)

Section 3

Proofs

Proof sketch

Suppose $f\colon \binom{[n]}{n/2} \to \{-1,1\}$ is ϵ-close to a linear function:

$$f(x_1,\ldots,x_n) \approx \sum_{i=1}^n c_i x_i =: \ell.$$

(Recall $\sum_{i=1}^n x_i = 0$.)

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3. Reduce to the case $x_i \approx 0$ for all i.

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2. $x_i \approx \pm 1$ for at most one i.
3. Reduce to the case $x_i \approx 0$ for all i.
4. Apply Friedgut–Kalai–Naor.

Applying Friedgut–Kalai–Naor

A *subcube* is a subset of the slice of the form

$$\{a_1, b_1\} \times \cdots \times \{a_{n/2}, b_{n/2}\}.$$

Corresponding restriction of ℓ is

$$g(y_1, \ldots, y_n) = C + \frac{1}{2} \sum_{i=1}^{n/2} (c_{a_i} - c_{b_i}) y_i.$$

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Friedgut–Kalai–Naor over a random subcube implies

$$\frac{n}{2} \sum_{i,j=1}^{n} (c_i - c_j)^2 = O(\epsilon).$$

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Left-hand side upper bounds the variance of ℓ.
Since $\mathbb{V}[\ell] = O(\epsilon)$, $\mathbb{E}[\ell] \approx \pm 1$. We are done since $f \approx \ell$.

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2. Let $h = \sum_{ij} b_{ij} x_{ij}$.
3. $\mathbb{E}[h^2] \approx \frac{1}{n} \sum_{ij} b_{ij}^2$, $\mathbb{E}[h^3] \approx \frac{1}{n} \sum_{ij} b_{ij}^3$.

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4. Since $f \approx \ell$, can estimate $\mathbb{E}[h^2] \approx c/n$.
5. Since $h \gtrsim 0$ and h is close to Boolean, $\mathbb{E}[h^3] \gtrsim c/n$.

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5. Since $h \gtrsim 0$ and h is close to Boolean, $\mathbb{E}[h^3] \gtrsim c/n$.
6. So $\sum_{ij} b_{ij}^2 (1 - b_{ij}) \lesssim 0$.

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7. So for each i,j, either $b_{ij} \approx 0$ or $b_{ij} \approx 1$.

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7. So for each i, j, either $b_{ij} \approx 0$ or $b_{ij} \approx 1$.
8. Roughly c of the b_{ij} are close to 1.

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7. So for each i,j, either $b_{ij} \approx 0$ or $b_{ij} \approx 1$.
8. Roughly c of the b_{ij} are close to 1.
9. $\mathcal{F}$ is approximated by union of the corresponding T_{ij}.

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Suppose $f\colon S_n \to \{-1,1\}$ is balanced ($\mathbb{E}f = 0$) and close to its linear projection ℓ.

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1. Let $a_{ij} = \frac{n-1}{n!}|\mathcal{F} \cap T_{ij}|$, so $f(\pi) = \sum_{i=1}^n a_{i\pi(i)}$.

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2. For random X, Y, consider $f|_{\pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}})$.

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2. For random X, Y, consider $f|_{\pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}})$.
3. For most X, Y, $f|_{\pi(X)=Y}$ close to Boolean ("(X,Y) good").

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3. For most X, Y, $f|_{\pi(X)=Y}$ close to Boolean ("(X,Y) good").
4. Since g_1, g_2 are "independent", one must be ≈ 0, the other close to Boolean.

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3. For most X, Y, $f|_{\pi(X)=Y}$ close to Boolean ("(X, Y) good").
4. Since g_1, g_2 are "independent", one must be ≈ 0, the other close to Boolean.
5. For most $\pi \in S_n$, most pairs $(X, \pi(X))$ are good.

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6. So $g_1(\pi|_X) = \sum_{i \in X} a_{i\pi(i)} \approx 0, 1$ (or $0, -1$) w.p. ≈ 1.

Proof sketch

Suppose $f\colon S_n \to \{-1,1\}$ is balanced ($\mathbb{E}f = 0$) and close to its linear projection ℓ.

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2. For random X, Y, consider $f|_{\pi(X)=Y} = g_1(\pi|_X) + g_2(\pi|_{\overline{X}})$.
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9. f essentially depends only on this strong row or column.